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Commentary

Auerbach's legacy

It is evident that throughout the world there are many towns, fewer large cities, and very few metropolises. This observation is not coincidental, but in effect follows a pattern and is often described by a power-law size distribution: that is, the probability of finding a city of size S is proportional to $S^{-\zeta}$ with ζ being close to 2. Today, this statistical regularity is referred to as Zipf's law for cities, being commonly attributed to the linguist and philologist George Kingsley Zipf, who originally studied the frequency of words in written texts, where he found an analogous distribution (Zipf, 2012). However, many researchers are not aware of the fact that the regularity of city sizes was described decades before by the theoretical physicist Felix Auerbach (1913). Zipf himself wrote "The first person to my knowledge to note the rectilinear distribution of communities in a country was Felix Auerbach in 1913" (2012, page 374). This year marks the centenary jubilee of this ground-breaking publication—an opportunity to review the legacy of Auerbach's paper, which is in danger of sinking into oblivion.

Auerbach's "Law of population concentration" aroused considerable interest at the time of its appearance. An important work in which reference to his findings was made is the seminal book on *Central Places* published by the geographer Walter Christaller in 1933. In his theory, cities are classified according to their centrality in terms of their excess of goods and services, balancing the insufficiency in the surroundings, and thereby providing supply. Spatially, the cities are arranged in hexagonal meshes on hierarchical levels with a dense grid of small cities and a sparse one of large cities. In today's terminology, the connection between the two hypotheses can be reduced to their scale-free nature: that is, the self-similarity of Christaller's model and the power law which constitutes Auerbach's size distribution.

Figure 1. Felix Auerbach during his time at the Institute for Theoretical Physics at the University of Jena (around 1925). From wikimedia, CC-BY-SA-3.0. Source: "Professorengalerie der Phys.-Astro. Fakultät". Author: Hartung, original uploader was PaulTTS.

Brian Berry (2012), another geographer and dedicated urbanologist, distinguishes between different phases of research on city-size distributions. The first period was initiated by Auerbach's discovery, followed by some publications elaborating on alternative representations. Mark Jefferson (1939) proposed the "The law of the primate city". Zipf's work, including his book *Human Behavior and the Principle of Least Effort* (1949), was the starting point of the second phase. This epoch gave rise to a flurry of studies estimating the exponent for city distributions and cross-national analyses (Carroll, 1982). John Quincy Stewart (1947) was another physicist studying urban features. Herbert Simon (1955) proposed a statistical process which is known today as preferential attachment, in which the probability of a city growing by a unit is proportional to its size. The lack of an intuitive theoretical model led to the third phase. Despite a variety of early modeling approaches from various disciplines, Berry attributes the first economic theory addressing city-size distributions to the economist Xavier Gabaix (1999). His approach involves size-independent growth rates, an assumption which is also known as Gibrat's law (Gibrat, 1931). Variant formulations were proposed in the subsequent years.

City-size distributions are still within the focus of contemporary research, with two main questions being studied. On the one hand, there is an ongoing dispute about whether city sizes indeed follow a power-law distribution or whether they are better described by another form, such as a log–normal. Even in case of the power law, the value of ζ is still being discussed and the causes of deviations are poorly understood. On the other hand, the mechanisms behind Auerbach's law are being investigated. Cities exhibit more properties than simply their sizes. Since multiplicative models, like that mentioned above (Gabaix, 1999), are zero dimensional, treating cities as units without location in space, they are not able to reproduce structural features such as fractality (Batty and Longley, 1994). Accordingly, a holistic theory is desirable, combining spatially explicit and economic approaches to produce a simple and universal city model. So far, no scientific consensus on either of these questions has been achieved.

The discourse gained new impetus when the definition of cities was questioned (Berry and Okulicz-Kozaryn, 2012). Administrative boundaries reflect the historical expansion of cities, and are often arbitrary. It has been shown that an objective definition based on the geographic circumference of cities—that is, as maximally connected urban areas—confirms Auerbach's law to an unprecedented extent (Rozenfeld et al, 2011). Another relatively new stream of city research addresses allometry (Bettencourt and West, 2010), that is, power-law correlations of certain socioeconomic indicators with city size, $Y \sim S^{\beta}$, implying, for instance, that large cities outperform small ones in terms of GDP per capita ($\beta > 1$). The obvious question of how Auerbach's law and the allometric scaling are related to each other is, to my knowledge, mostly unanswered. A starting point could be Christaller's central places theory.

Interestingly, Auerbach had already suggested in his 1913 paper that his law of city sizes was "only a special case of a much more general law which can be applied to the most diverse problems in the natural sciences, geography, statistics, economics, etc." (page 76). This claim is supported by the extensive evidence that is known today. Beyond cities and words, power-law distributions are found in seismology [Gutenberg–Richter law (Gutenberg and Richter, 1949)], sexual contacts (Liljeros et al, 2001), and many other fields (Newman, 2005). In all cases, similar distributions are found—only the quantities are different. Historically, the very first domain where the power-law distribution was described was income, set out in 1896 by the economist Vilfredo Pareto, after whom the general Pareto distribution was named (Pareto, 1896).

Auerbach was a notable personage in both academic and private life. Born in 1856 in Breslau, now Wrocław in Poland, at a young age he became a student of Hermann von Helmholtz and Gustav Kirchhoff in Heidelberg. In 1875 Auerbach finished his PhD under the supervision of Helmholtz, and in 1879 completed his habilitation treatise on harmony and melody, illustrating his interdisciplinary interests. His further research fields included magnetism and hydrodynamics. In 1889 he became a professor in Jena, where he subsequently obtained a full professorship in 1923, retiring in 1927. Auerbach had close contact with intellectuals and artists:

"Auerbach was a musician of high standing, an art lover, and a man who appreciated literature and made his own small contribution to it" (Kisch-Arndt, 1964).

He and his wife Anna were Jews, committing suicide in 1933 shortly after the Machtergreifung of the Nazis.

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