Mattauch, L., Klenert, D., Stiglitz, J. E., Edenhofer, O. (2022): Overcoming wealth inequality by capital taxes that finance public investment. - Structural Change and Economic Dynamics, 63, 383-395.

DOI: https://doi.org/10.1016/j.strueco.2022.05.009

Overcoming wealth inequality by capital taxes that finance public investment

Linus Mattauch* David Klenert† Joseph E. Stiglitz‡
Ottmar Edenhofer§

April 10, 2022

Abstract

Wealth inequality is rising in high-income countries. Can increased public investment financed by higher capital taxes counteract this trend? We examine how such a policy affects the distribution of wealth in a setting with distinct wealth groups: dynastic savers and life-cycle savers. Our main finding is that this policy always decreases wealth inequality when the elasticity of substitution between capital and labor is moderately high. At high capital tax rates, dynastic savers disappear. Below these rates, life-cycle savers gain from the higher public expenditures financed by the higher capital tax rates. We calibrate our model to OECD economies and find a threshold elasticity of 0.82.

JEL classification: D31, E21, H22, H31, H41, H54

Keywords: Public capital, capital taxation, wealth distribution, household heterogeneity, saving behavior

*Technical University of Berlin, Institute for New Economic Thinking at the Oxford Martin School, University of Oxford, Potsdam Institute for Climate Impact Research. E-mail: linus.mattauch@inet.ox.ac.uk

†Joint Research Centre of the European Commission

‡Columbia University

§Technical University of Berlin, Mercator Research Institute on Global Commons and Climate Change and Potsdam Institute for Climate Impact Research

1 Introduction

Wealth inequality is rising in rich countries (Piketty and Saez, 2014; Saez and Zucman, 2020). Taxing capital to counteract this trend is no panacea, as the burden of the tax often shifts to "workers", i. e. low- and middle-income households, who predominantly save for retirement. In studying tax incidence in general equilibrium, the effects of either substituting one tax with another, or of balancing the revenues with a corresponding increase in expenditures, has to be considered. But nonetheless, there is a widely held predilection to separate "expenditure" from "direct" effects: How the tax proceeds are allocated can clearly make a difference for incidence. In spite of the widespread advocacy of capital taxation to redress wealth inequalities (see Piketty, 2014), it appears that if the proceeds are spent in the way most seemingly favourable to workers, direct transfers, they would actually be worse off due to general equilibrium effects (Judd, 1985; Stiglitz, 2016b, 2018a). Expenditure programmes on public education can, by contrast, increase workers' wellbeing (Hanushek and Woessmann, 2008). Their net effect depends on how they are financed.

In practice, taxes are often linked with expenditures. Public capital, notably in education and infrastructure, increases productivity in the long term but is often underfunded (Bom and Ligthart, 2014; Calderón and Servén, 2014). One prominent policy proposal to reduce wealth inequality is hence to tax the returns to private capital and use the proceeds to increase public capital (Diamond and Saez, 2011; Piketty, 2014; Piketty and Zucman, 2014; Stiglitz, 2012, 2016a). In this paper, we examine the conditions under which this proposal would increase workers' wellbeing and reduce inequality, looking at the steady state of an overlapping generations model with life cycle and dynastic savers ("workers" and "capitalists").

Specifically, we look for the conditions in which the positive "expenditure incidence effect" of capital taxes outweighs the direct tax adverse incidence effect. We provide an unambiguous answer: a complete characterization of the possible long-run distributional outcomes depends on the tax rate, forms of productive public investment, and substitutability between capital and labor. We prove that, depending on the value of the elasticity of substitution between capital and labor σ, and on the level of the capital tax τ, there are three cases: (a) both classes can co-exist; (b) capitalists disappear, as their absolute income becomes zero; or (c) workers relative income goes to zero. A change in taxes can result in the economy moving from one regime to another, and we treat (b) and (c) as limiting cases that permit to characterize the impact of the policy on the case where classes co-exist. We analyze, in particular, the cases where a high enough tax on capital would effectively obliterate capitalists, so their share in national income goes to zero. That will always be the case, we show, if the limiting value of the elasticity of substitution (as the ratio of capital and effective labor goes to zero) is greater than unity. More generally, if the elasticity of substitution is greater than unity, an increase in the tax rate lowers wealth inequality.

Most of the analysis focuses on the possibility of mitigating wealth inequality by capital taxes in the regime where both classes co-exist. For a constant elasticity of substitution (CES) production function specifically, there exists a capital tax rate τ_{lim} so high that either capitalists disappear or workers' share of wealth goes to zero. The former result holds if the elasticity of substitution is above a given threshold, the latter below it. We also establish, by numerical simulations, that even for relatively low elasticities workers still gain in absolute and relative terms from moderate capital tax rates.

The intuition for these results is as follows: a tax on capital discourages capitalists from saving. In our model, in the regime in which there are both pure capitalists and workers (who engage in life cycle savings), capitalists' savings *alone* determine the long run capital labor ratio—it is the ratio such that the *after tax rate of return* equals their pure rate of time preference. Thus, workers' wages (at any level of human capital) fall; but if, say, the tax proceeds are invested in human capital the returns to which workers appropriate, they are better on that account. The impact on relative income shares of the increased supply of "effective" labor combined with the decreased savings by capitalists depends on the elasticity of substitution. If the elasticity of substitution is high enough—in our calibrated simulation, greater than .82—workers' share in income increases. We find that this also implies that their share of capital increases, indeed monotonically as the capital tax is increased, to the point where capitalists disappear. If the elasticity of substitution is below that threshold, but not by too much, then moderate capital taxation still reduces inequality. But if the elasticity of substitution is below a certain threshold, workers' shares essentially monotonically decrease as the tax increases.

The simulation also indicates, however, that even when workers' shares decrease, so long as taxes are moderate, they are still better off in absolute terms. Because of underinvestment in public capital, at least for low tax rates, an increase in the tax on the returns to private capital increases output, so that so long as workers' share does not diminish too rapidly (i.e. so long as the elasticity of substitution is not too small) workers are better off. It has previously been shown analytically that under restrictive conditions capital-tax financed public investment can be Pareto-improving (Mattauch et al., 2016; Klenert et al., 2018; Stiglitz, 2018a), while numerically verified that financing it by labor- or consumption taxes does not decrease wealth inequality (Klenert et al., 2018). In more general settings, however, the concern prevails that the burden of capital taxes may still be shifted to the workers, thus increasing wealth inequality (Stiglitz, 2016a), which we examine in this contribution.

The elasticity of substitution could, of course, differ across different countries, implying that a policy appropriate for one country might not be for another. Our review of the plethora of estimates of the elasticity of substitution suggests that capital taxation can be an effective tool for redistribution *if* the proceeds are used to finance public investment; it is not the case only for values of the elasticity of substitution that are smaller than those typically found in the literature.[1]

The next section relates our contribution to earlier literature. Section 3 sets out the model. In Section 4, we characterize wealth inequality for general production functions. In Section 5, we derive results with a CES production function in which public capital is labor-enhancing and in Section 6, we consider alternative specifications of public capital, including cases where it acts on labor differently or is capital-enhancing. Section 7 presents our numerical application. Section 8 concludes and outlines how increasing automation reinforces the policy implications of our results.

2 Earlier literature

There is, of course, a large body of literature on the incidence of capital taxation and its usefulness, especially in the long run, as a tool in achieving equalitarian objectives. As we previously noted, much of the earlier literature was not sanguine: In a model similar to the one presented here, the substitution of a capital tax for a (lump-sum) tax on workers made workers worse off, because of adverse impacts on wages (see e. g. Stiglitz (2016a, 2018a)). Others, including Judd (1985) have argued that optimal taxation requires, in the long run, a zero capital tax. Atkinson and Stiglitz (1976)'s analysis of optimal non-linear taxation in the presence of separability between leisure and consumption goods seemed to imply that optimality required no taxation of the returns to capital. A more careful analysis, within that framework, shows

[1] There is a recent debate in the empirical literature over the value of the elasticity of substitution. For example, Chirinko (2008) show that 26 out of 31 studies find an elasticity of substitution between capital and labor (significantly) below 1. By contrast, Piketty and Saez (2014) and Piketty and Zucman (2015) argue that the elasticity must be higher than 1. In particular, Piketty and Saez (2014) argue that "it makes sense to assume that σ tends to rise over the development process, as there are more diverse uses and forms for capital and more possibilities to substitute capital for labor." (p. 841). But this argument is not fully persuasive, because advances in technology can result in dominating technologies, leading to a lower elasticity of substitution (Atkinson and Stiglitz, 1969; Korinek and Stiglitz, 2017). In any case, the discussion of the aggregate elasticity of substitution entails delicate issues of capital aggregation (Stiglitz, 2015b).

that that is not the case, see Stiglitz (2018a) for an extensive discussion.

Within this large literature on capital taxation and inequality, our contribution is the first to analyze systematically how the success of capital taxes financing public investment in terms of addressing inequality depends on the elasticity of substitution. Broadly speaking, there are three discernible strands of literature on capital taxation and inequality: (i) households suffer idiosyncratic uninsurable earnings or productivity shocks and have a precautionary savings motives, but are otherwise only mildly heterogenous (Bewley, 1977; Aiyagari, 1994); (ii) heterogeneity in wealth is generated from life cycle savings, with different individuals having different wages and wage profiles, with some variants focused on with approximating the upper tails of the income and wealth distribution well (Champernowne, 1953; Stiglitz, 1966; Gabaix, 2009; Jones, 2015); (iii) heterogeneity in wealth is generated from inheritances, possibly with stochastic returns, and with differential wages, with abilities passed on across generations, possibly with regression towards the mean (Stiglitz, 1966, 1978; Bevan, 1974; Bevan and Stiglitz, 1979). In each of these models, one can examine how capital, inheritance and wealth taxes changes distribution and output. Here, we combine a model with life cycle savings and long term dynastic savings, focusing on the distribution of wealth between these two groups, rather than the distribution of wealth within either group. It is a straightforward matter to incorporate within our model heterogeneity (at least of particular forms) within each of the two "classes."[2] Some models assume heterogeneity in initial capital endowments only and then have to assume endogenous growth in order for the households not to converge (Chatterjee and Turnovsky, 2012), see also Becker (1980). However, distinguishing heterogeneous groups of savers provides an alternative to these models in which a sustained long-term wealth distribution is possible.

An early model which can be thought of as an antecedent of that presented here, is that of Pasinetti (1962). There, workers save a fixed fraction of their income and capitalists of theirs. This model has has been taken up by Samuelson and Modigliani (1966), Stiglitz (1967) Judd (1985), Baranzini (1991), Michl (2009) and Straub and Werning (2020). Meade (1966) and Samuelson and Modigliani (1966) noted that if workers' savings rates were large enough relative to that of capitalists, capitalists would not survive in the long run. Samuelson and Modigliani (1966) labelled the case in which capitalists disappear "Anti-Pasinetti regime", and this earlier literature clarified how it is a special case of the co-existence of workers and capitalists.

[2] Without savings and wage heterogeneity (whether inherited or simply "noise"), Stiglitz (1969) showed that there would be wealth convergence, even if initially individuals had different levels of wealth. Stiglitz (1966, 1969) shows how various forms of heterogeneity can give rise to steady state distributions. See Stiglitz (2021) for a fuller exposition.

While these papers did not derive savings from optimal intertemporal maximization, it is easy to do so, as we show below. Judd (1985) consider the case in which workers save nothing, his paper centers around that in which all individuals are identical (and have homothetic preferences) except for their initial value of wealth. In the latter case, he obtains the strong result that it is optimal in the long run to have a zero capital tax. Straub and Werning (2020) and Stiglitz (2018a) recently have shown that this stark result is not general; for instance, it does not necessarily hold with more general utility functions. Our approach differs by explicitly allowing for heterogeneity in preferences, with a model in which there are both life cycle and dynastic savers, focusing on the distribution of wealth between these two groups and how it is affected by capital tax-financed public investment and elasticities of substitution. We show that such preference heterogeneity can be consistently analyzed in a general equilibrium production framework in which public capital is a separate production factor. Franks et al. (2018) is a related contribution, studying several intermediate classes that display a mixture between bequest and life-cycle motivated saving and shows that it is more efficient to tax certain components of aggregate wealth, namely land rents or bequests than aggregate capital. This is a feature from which the present analysis abstracts, as does most of the literature on capital taxation, such as Piketty and Saez (2013).

3 The basic model

Guiding our analysis is an attempt to formulate a model that is consistent with the "new" stylized facts regarding growth and distribution (see, e.g. Stiglitz (2015b)), which replace the Kaldor facts that were integral in the development of the neoclassical growth model. These stylized facts include: (a) growing inequality in both wages and capital income (wealth), and growing inequality overall[3]; (b) wealth being more unequally distributed than wages; (c) average wage stagnation, (d) increases in the wealth–income ratio; (e) the return to capital not having declined, as the wealth–income ratio has increased (Stiglitz, 2016b). Growing empirical evidence further suggests that individuals at the top of the wealth distribution display saving behavior that is markedly different from other households: Rich individuals have higher saving rates, obtain a greater share of their income from capital and save more for posterity rather than for retirement (Attanasio, 1994; Dynan et al., 2004; Lawrance, 1991; Saez and Zucman, 2016; Epper et al., 2020). Models should thus account for heterogeneous preferences (Krusell and Smith, 1998;

[3]For evidence that the wealth distribution is more skewed than the labor income distribution; and rising top income and wealth shares in nearly all countries in recent decades see also Alvaredo et al. (2017); Novokmet et al. (2017); Wolff (2017).

Foley and Michl, 1999), especially with respect to saving behavior (Stiglitz, 2015b, 2016b).

To address these realities, in a simple framework that allows for inequality between households even in the long run, we consider how increased public investment that is financed by capital taxes affects wealth inequality. We combine the two standard approaches to saving behavior—the dynastic model and the life-cycle model—in a single framework. This simplification captures some of the stylized facts by representing the saving behavior of most citizens as occurring within their "life cycle", while representing a second group of citizens with such high levels of inherited wealth as to make their labor income irrelevant and their saving behavior dynastic. The model thus represents the above stylised facts on growing inequality in wage and capital income (a), wealth being distributed more unequal than wages (b) and, to an extent, increases in the wealth-income ratio (d).[4] The model does not seek to explain the origins of those class differences, but permits us to study a key feature of wealth inequality and the policies that might address it. Stiglitz (2015d) shows this simplification is a limiting case of a model in which households with highly non-linear savings functions can transition between (endogenous) wealth groups.[5]

Specifically, our model consists of an economy with a single consumption good in which the government can finance productivity-enhancing public investment by a capital tax. "Workers" live for two periods. They receive income from labor when young and save for retirement (broadly understand, including private savings), during which they receive capital income. "Capitalists", the top wealth owners, have a dynastic saving motive and are modelled as identical representative infinitely-lived agents. They receive only interest payments on their capital holdings and, in some cases, firms' profits.Berman and Milanović (2020) and Ranaldi and Milanović (2022) recently collect evidence indicating that this distinction is necessarily a simplification:

[4]Stylised fact (c) is often attributed to significant market power (see Stiglitz (2016b)). The most promising explanation behind the lack of a declining capital return (e) (and influencing (d)) is that wealth is in fact composed of productive capital and fixed factors such as land, see Edenhofer et al. (2015); Franks et al. (2018); Franks and Edenhofer (2020); Mattauch et al. (2018); Stiglitz (2015e, 2016a,b).

[5]The differences in savings rates could just reflect large differences in wealth, i. e. if "workers" were given a large transfer of money, they might have higher savings rates and give large bequests. The framework could hence be extended to include idiosyncratic shocks and wage differentials–as pioneered by (Aiyagari, 1994; Bewley, 1977; Bevan, 1974, 1979; Stiglitz, 1966, 1969, 1978, 2021) and which matter for a richer picture of wealth inequality. The present contribution can therefore be seen as part of a broader research strategy in which idiosyncratic shocks could be introduced and the structure of the economy–between life-cycle and dynastic savers–is itself endogenous. It is worth noting that a significant fraction of the very wealthy in the US are, to a significant extent, "self-made"; that is their wealth is not based on inheritances.

an increasing proportion of those capital-income rich are also labor-income rich .[6] In the model, we assume throughout that workers are poorer than capitalists. It is assumed merely for analytical simplicity in the next sections that capitalists have no labor income. Furthermore, factor markets clear and on the capital market, the supply consists of both agents' capital holdings. We examine the distribution of wealth, but since consumption is linear in wealth in our basic model for both groups and utility only depends on consumption, results qualitatively carry over to the distribution of consumption and utility.

Capitalists The capitalists owns a capital stock K_t^c and maximize intertemporal utility given by

$$\sum_{t=0}^{\infty} \frac{1}{(1+\rho_c)^t} \ln(C_t^c), \tag{1}$$

with consumption C_t^c and time preference rate ρ_c., subject to the budget constraint

$$K_{t+1}^c - K_t^c = (1-\tau) r_t K_t^c - C_t^c + \Pi_t, \tag{2}$$

where r_t is the before-tax interest rate. A capital income tax τ is imposed on all capital. Firms' profits Π_t may be zero, depending on the production structure. The initial capital stock is given as K_0^c. The capitalist respects a transversality condition: $\lim_{t\to\infty}\left(K_t^c \prod_{s=1}^{t-1} \frac{1}{1+r_s}\right) \geq 0$.

Solving the maximization problem yields an Euler equation for this household:

$$\frac{C_{t+1}^c}{C_t^c} = \frac{1+(1-\tau) r_{t+1}}{1+\rho_c}. \tag{3}$$

Workers The worker lives for two periods, a "young" (y) and an "old" (o) stage. It maximizes its lifetime utility, with utility from consumption in the second period being discounted by the time preference rate ρ_w:

$$\ln(C_t^y) + \frac{1}{1+\rho_w} \ln(C_{t+1}^o). \tag{4}$$

In the first period, the agent sells its fixed labor L to the producing firm, which in turn pays a wage rate w_t. Labor income can either be consumed or saved for the old age:

$$w_t L = S_t + C_t^y. \tag{5}$$

[6]Individuals at the top of the wealth distribution include self-employed entrepreneurs but who receive a higher share of capital income (Wolff, 1998; Diaz-Gimenez et al., 2011; Wolff and Zacharias, 2013).

In the second period the agent consumes its savings and the interest on them:

$$C^o_{t+1} = (1 + (1 - \tau)r_{t+1})S_t. \tag{6}$$

Solving the optimization problem subject to the budget constraints leads to an Euler equation for this household:

$$\frac{C^o_{t+1}}{C^y_t} = \frac{1 + (1 - \tau)r_{t+1}}{1 + \rho_w}. \tag{7}$$

From Equations (5-7) an explicit expression for saving can be derived:

$$S_t = \frac{1}{2 + \rho_w} w_t L. \tag{8}$$

This implies a constant savings rate of $1/(2+\rho_w)$, as is standard in discrete OLG models when the utility function is logarithmic.[7]

Production Consider a production sector given by the production function $F(P_t, K_t, L)$, with P_t public capital. K_t denotes the sum of the individual capital stocks

$$K_t = K^c_t + S_{t-1}. \tag{9}$$

Throughout we assume constant returns to scale in all three factors: $F(P_t, K_t, L) = F_K K + F_L L + F_P P$. Later sections will focus on the case where the production function is CES.

Government The sole function of the government in this model is the provision of public capital. It finances its investments using the capital income tax revenue, thus influencing the interest rate. Hence the government's activity is summarized as the change in the stock of public capital (with δ_P denoting its depreciation):

$$P_{t+1} = (1 - \delta_P)P_t + \tau r_t K_t. \tag{10}$$

Return to public investment There are a number of ways to close the model by specifying how the return to public investment is distributed to the agents and whether and how it modifies returns to the other production factors, with different economic interpretations (see also Section 6). For the

[7]Thus, standard Keynesian style models that begin with a constant savings rate can easily be provided micro-foundations. Note however that modern behavioral economics suggests that it may be misguided to demand such micro-foundations, see Stiglitz (2018b). Note too that savings rates will depend on the real interest rate when the utility function is not logarithmic, but the equilibrium can still be analyzed with techniques similar to those presented here, see Stiglitz (2018a).

basic model, we focus on the case of public capital as investment into education, enhancing the productivity of labor. For this case, assume that workers' enhanced labor is a constant-returns-to-scale sub-production function J

$$J(P_t, L) = LJ(P_t/L, 1), \tag{11}$$

a function of the labor supply and education expenditures. Without loss of generality, we normalize the labor supply at unity for this section and the next and define $j(P) \equiv J(P_t, L)$. Total production is then given by $F(K, J)$ and is constant-returns-to-scale in K and J. Further, $f(k) = F(K, J)/J$ as usual and let $\sigma(k)$ be the elasticity of substitution between K and J defined via f. With this specification, the functions F and J, in combination with the parameters ρ_w and ρ_c determine the equilibrium—including the equilibrium distribution of wealth. It is natural to define w as the wage per efficiency unit of labor, J :

$$w_t = \frac{\partial F(K_t, J_t)}{\partial J}. \tag{12}$$

and total wage income is $w_t J_t$. Workers appropriate all the return to labor, so the budget constraint of workers becomes

$$w_t J_t = S_t + C_t^y, \tag{5a}$$

so that

$$S_t = \frac{1}{2 + \rho_w} w_t J_t. \tag{8a}$$

Profit maximization of the firm yields the standard rates of return to capital (with δ_K denoting depreciation of private capital):

$$r_t + \delta_K = \frac{\partial F(P_t, K_t, L)}{\partial K_t}. \tag{13}$$

Profits in Equation (2) are set to zero as a consequence. We employ this version of the model in the following unless stated otherwise.

4 Results for general production functions

In this section, we determine the basic properties of the model for general production functions, focusing on the case where tax revenues are allocated to human capital. Analytically, we focus on relative improvements, i. e. changes in the distribution of wealth, for different uses of the tax revenue, in general equilibrium. See Appendix A for translating prior results on absolute improvements into our setting. The model is solved for steady states for general production functions; steady-state values of variables are denoted by a tilde.

We focus on the steady-state equilibria which emerge if K_c initially is greater than zero.[8] It follows from the capitalist's Euler Equation (3), in any equilibrium with a steady-state capitalist that the steady-state interest rate $\widetilde{r}$ is given by

$$\widetilde{r} = \frac{\rho_c}{(1-\tau)}. \tag{15}$$

This means that the steady-state interest rate is solely determined by the capitalists' time preference rate and if there is a steady state with capitalists, then there is full shifting of capital taxation (see Pasinetti (1962) and Stiglitz (2016a)).[9] From (10) in steady state

$$\widetilde{P} = \frac{1}{\delta_P}\tau\widetilde{r}\widetilde{K} = \frac{1}{\delta_P}\tau\frac{\rho_c}{(1-\tau)}\widetilde{K}. \tag{16}$$

That is, for a given capital tax rate, public capital is just proportional to private capital. Equations (15) and (16) determine the allocation of total private and public capital in the economy, with the share of public capital increasing from 0 when $\tau = 0$.

The share of workers' wealth for a general production function can be determined by dividing Equation (8a) by total capital:

$$\frac{\widetilde{S}}{\widetilde{K}} = \frac{1}{2+\rho_w}\frac{\widetilde{w}\widetilde{J}}{\widetilde{K}}. \tag{17}$$

The above equations only hold, of course, if $\widetilde{S} < \widetilde{K}$. In the following, we use the share of workers wealth in total wealth as a measure of wealth inequality, that is we assume throughout that workers are poorer than capitalists. Three cases can result in the long term. Both classes can co-exist ("Pasinetti-regime"), but at the margins of the model the capitalists disappear as their absolute income becomes zero ("Anti-Pasinetti-regime") or workers' savings become zero as labor disappears in relative terms in the limit ("Anti-Anti-Pasinetti-regime"). Much of our analysis concerns examining the location of

[8]If capitalist initial wealth is zero, then capitalists remain in such a steady state. The model becomes a variant of the standard overlapping generations model. Corresponding to any value of τ, there is a steady state equilibrium given by the solution to

$$\widetilde{K} = \frac{1}{2+\rho_w}\widetilde{w}(\widetilde{K})\widetilde{J}(\tau\widetilde{K}). \tag{14}$$

[9]It is no surprise that we obtain consequences similar to Pasinetti (1962), because under our assumptions, savings rates are effectively fixed. More precisely, if profits are zero, Equations (2), (6) and (7) imply that all consumption variables are linear in wealth for the two groups, but with different factors. We have simply provided the obvious micro-foundations.

this boundary, which is simply the value of τ for which $\frac{\widetilde{S}}{\widetilde{K}} = 1$. The Anti-Pasinetti regime, in which only workers exist (formally given by a wealth ratio of $\widetilde{S}/\widetilde{K} = 1$) yields a standard discrete overlapping generations model with public capital (Heijdra, 2009, Ch. 17). Taylor (2014) and Zamparelli (2017) discuss the economic significance of the case in which workers disappear, the Anti-Anti-Pasinetti regime (see Baranzini and Mirante (2021)), while the subsequent discussion treats both as a limiting cases. The analysis of behaviour at the boundary as a simple way to characterise formally how inequality changes with the policy in the "Pasinetti-regime". We provide a sufficiency condition for workers to be absolutely better off in Appendix B.1.

It is possible to characterize the limiting behavior of the model by making assumptions about the elasticity of substitution between capital and (augmented) labor and the factor shares. Define factor shares Ω_J and Ω_K, noting labor is augmented by public investment to yield the composite J:

$$\Omega_J = \frac{F_J J}{Y} \qquad \Omega_K = \frac{F_K K}{Y}. \tag{18}$$

From Equation (17) one obtains

$$\frac{\widetilde{S}}{\widetilde{K}}(\tau) = \frac{1}{2+\rho_w}\widetilde{\Omega}_J(\tau)/\widetilde{\Omega}_K(\tau)\Big(\frac{\rho_c}{1-\tau} + \delta_k\Big) \tag{19}$$

It is immediate that since $\rho_c/(1-\tau)$ tends to infinity as $\tau \to 1$, if the factor share accruing to enhanced labor is strictly positive as $\tau \to 1$, then $\frac{\widetilde{S}}{\widetilde{K}}(\tau)$ exceeds 1 (and in fact diverges). If for some tax rate both classes exist, then there will be a tax rate at which capitalists vanish, by the Intermediate Value Theorem.

Keeping everything else fixed, an increase in workers' saving propensity lowers capitalists' share in wealth. If the elasticity of substitution is equal to one, shares are fixed, so it is clear that an increase in ρ_c (an increase in capitalists' time discounting, inducing them to save less) also leads to a larger workers' share of capital, as does an increase in the capital tax rate. If the elasticity of substitution is greater than unity, an increase in the capital tax rate lowers κ and hence increases the share of labor, so the indirect effect reinforces the direct effect, and the share of capital owned by workers is increased. But if the elasticity of substitution is *enough* below unity, the effects could be reversed. The effective capital-labor ratio must decrease to restore the after tax return to ρ_c, and if the elasticity of substitution is low enough, that so lowers the share of labor that S/K is reduced. A striking feature of this result is that it does not depend on properties of J, only on those of F. Properties of J determine how changes in κ translate into changes in K, and thus P and workers incomes. We return to these properties of the model for specific parametrisations in Sections 5 and 6. Here, one can establish that more generally:

Proposition 1. *Assume* $0 < \frac{\widetilde{S}}{\widetilde{K}}(\epsilon) < 1$, *i.e. for small taxes rates* $\epsilon > 0$ *both classes coexist. Further assume* $\lim_{\tau \to 1} k(\tau) = 0$. *If* $\sigma(0) > 1$, *with* $\sigma(0)$ *the limiting elasticity of substitution as* $\tau \to 1$, *then there always exists a capital tax rate* $\tau_{\lim}$ *such that capitalists vanish (Anti-Pasinetti case).*

Proof. It is known that $\sigma(0) > 1$ implies $\lim_{k \to 0} \widetilde{\Omega}_K(\tau) = 0$ (see for instance Barelli and de Abreu Pessôa, 2003). Thus Equation (19) diverges to infinity as $\tau \to 1$. Therefore, the conclusion follows from the Intermediate Value Theorem. □

Later, we examine specific production functions, for which the condition is satisfied. When $\sigma(0) < 1$ instead, the above argument does not hold because it does not lead to convergence of Equation (19) to a value less than 1.

Using these results, we can moreover ascertain what happens to capitalists' steady state income, $Y_c = \rho_c(K - wj/(2 + \rho_w)$. Note that

$$dY_c/d\tau = d(\rho_c K(1 - S/K))/d\tau. \tag{20}$$

For small τ, we have established that K increases with τ if $\eta > 1$. Also, if the elasticity of substitution σ between capital and augmented labor is sufficiently small, S/K decreases. Hence if η is large enough and σ is small enough, capitalists' income increases, too.

Next, we characterize the boundary of the regime in which both classes co-exist for the case of public investment as education. Proposition 1 merely states that a tax rate exists at which capitalists eventually vanish: the number of switches between regimes cannot be concluded from it. Let $\Phi(\tau) = \frac{\Omega_J(\tau)}{\Omega_K(\tau)}$. At the boundary

$$1 = \frac{1}{2 + \rho_w} \Phi(\tau) \Big(\frac{\rho_c}{1 - \tau} + \delta_k \Big). \tag{21}$$

Any value of τ for which the above equation is satisfied is a switch-line. If $\Phi'(\tau) > 0$ there is a unique solution in τ. But more generally there can be multiple solutions to Equation (21). We can prove Proposition 2 (see Figure 1 for an illustration.[10]):

Proposition 2 (General characterization of Pasinetti-regime boundary)**.** *Suppose* $0 < \frac{\widetilde{S}}{\widetilde{K}}(\epsilon) < 1$, *i.e. for small taxes rates* $\epsilon > 0$ *both classes coexist. Suppose that* $\Phi(\tau)$ *is a monotone, continuous function on* $(0, 1)$.

[10]Note that in Equation (19) values above 1 are not economically meaningful, but still, the left-hand side of is well-defined—we simply need to ascertain the values of τ for which its value is smaller than 1.

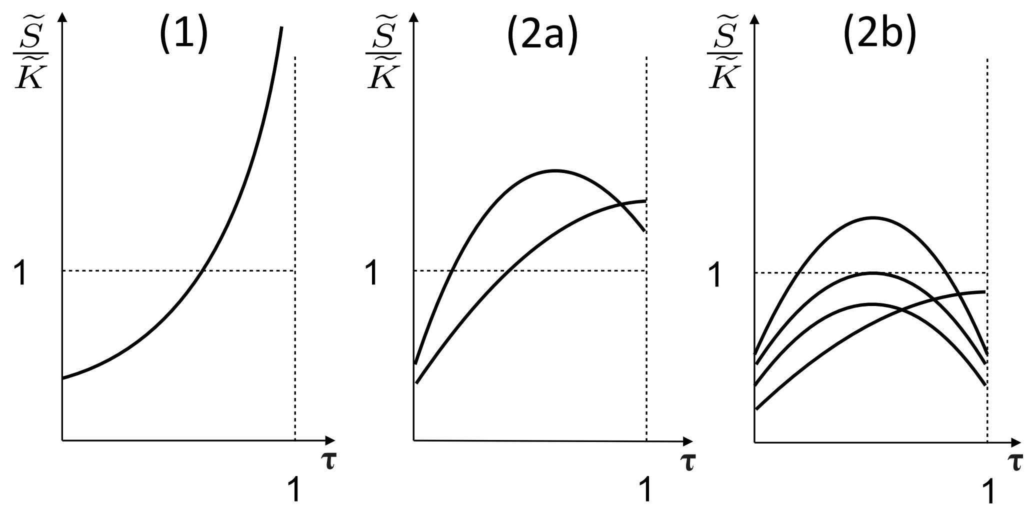


Figure 1: Schematic illustration of the possibilities for the Pasinetti-regime boundary occuring in the cases of Proposition 2.

1. *Suppose Φ is increasing. Then only one switch from the Pasinetti regime to the Anti-Pasinetti regime occurs.*

2. *Suppose instead that Φ is decreasing. Suppose further that Φ satisfies the following inequality:*

$$\frac{1}{2+\rho_w}\Phi''(\tau)\left(\frac{\rho_c}{1-\tau}+\delta_k\right)+2\Phi'(\tau)\left(\frac{\rho_c}{(1-\tau)^2}\right)+\Phi(\tau)\frac{2\rho_c}{(1-\tau)^3}<0 \tag{22}$$

   Then

   (a) *if $lim_{\tau\to 1}\frac{\widetilde{S}}{\widetilde{K}} > 1$, only one switch from the Pasinetti to the Anti-Pasinetti regime occurs.*

   (b) *if $lim_{\tau\to 1}\frac{\widetilde{S}}{\widetilde{K}} \leq 1$, either the Pasinetti regime persists for all tax rates, or there may be a switch from the Pasinetti to the Anti-Pasinetti regime for some tax rate followed by a switch back for a higher tax rate (which can coincide).*

Proof. See Appendix B.2. □

Finally, for a marginal change in the tax rate, it is also possible to characterize the effect of an increase in the tax rate on relative capital holdings in terms of the elasticity of substitution $\sigma(k)$ for general production functions, in a two class equilibrium, see Appendix B.3.

5 Labor-enhancing public investment

This section extends the analysis of the previous one by parametrizing the production function by a CES function between capital and effective labor, and in which public capital P is labor-enhancing (as in the previous section):

$$F(P, K, L) = \left(\alpha K^\gamma + (1-\alpha)(J)^\gamma\right)^{\frac{1}{\gamma}} \tag{23}$$

with $0 < \alpha < 1$, $\gamma < 1, \gamma \neq 0$.[11] The elasticity of substitution between capital and labor σ is given by $\sigma = 1/(1-\gamma)$. The relative capital intensity at any given wage-interest-ratio is reflected by α. It is an illustration that provides more intuition for the abstract general results.

The results are illustrated by Figure 2. Throughout this section we assume that a steady state in relevant ranges of the Pasinetti regime exists.[12]

We derive in Appendix D.1 that

$$\frac{\widetilde{S}}{\widetilde{K}}(\tau;\gamma) = \frac{1}{\alpha(2+\rho_w)}\Big(\frac{\rho_c}{1-\tau} + \delta_k\Big)\Big((\frac{1}{\alpha}(\frac{\rho_c}{1-\tau} + \delta_k))^{\frac{\gamma}{1-\gamma}} - \alpha\Big). \tag{24}$$

Straightforward computation shows that this expression is monotonically decreasing in α. This means that wealth inequality increases with higher capital intensity, as is to be expected (capitalists derive all their income from capital; workers only a fraction of their income). In this section, we focus on how substitution elasticities effect the impact of capital taxation in the long run.[13]

Assumption 1. *(a) For a capital tax of nearly zero both agents co-exist. This implies that* $0 < \widetilde{S}/\widetilde{K}(\epsilon, \gamma) < 1$ *for* $\epsilon > 0$ *small.*

[11] To avoid a confusion in units, as K is measured in capital goods while J in equivalent labor units, one should, strictly speaking, account these different goods in "aK" and "b," where a and b are such as to ensure equivalency of services provided, i.e. so that if K is the only factor of production, $Q/K = a$, and similarly for J. We choose our units so that a = b = 1.

[12] It is well-known that for CES functions, a steady state might not exist for all values of the elasticity in a neoclassical growth model because for each value of the elasticity, one of the Inada conditions is violated. In this section, existence would also depend on the function J, so we assume it is such that a steady state exists with both classes (the Pasinetti regime) for a capital tax of zero. In Appendix C we show that there are plausible specifications of J for which this is true. Further, whenever a unique steady state exists, the model converges to it because it inherits the dynamics of a neoclassical growth model with public capital, see Appendix C.

[13] Changing only one of the two parameters of a CES function, as we explore in this section, changes the distribution of income at any given ratio of capital and (effective) labor. The net effect on the equilibrium if we change both parameters simultaneously so as to preserve the distribution of income in the initial situation would be different from that when we perturb only one parameter.

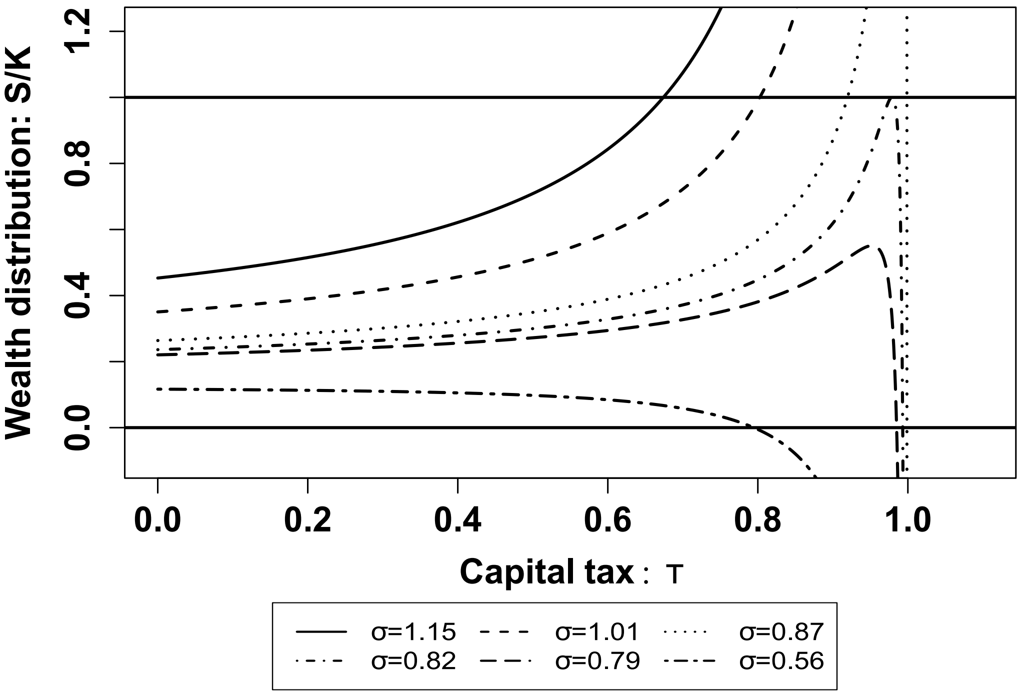


Figure 2: Wealth inequality as a function of the capital tax rate for various elasticities as an illustration of Proposition 3, 4 and Corollary 5 (plotting Euqation 24). For a discussion of our model calibration see Section 7.

(b) $\delta_K > \alpha$.

Both of these assumptions hold for the economically relevant range of the parameters used in our model by a very large margin.[14]

First, consider the case that $\gamma > 0$, that is, the substitution elasticity between capital and labor is greater than 1.

Proposition 3. *Let production be specified as above and assume* $\gamma \geq 0$.

(a) For every γ, *there exists a capital tax rate* τ_{lim}, *such that capitalists vanish, that is, the solution to Equation* (24) *entails* $\widetilde{S}/\widetilde{K} = 1$ *(Anti-Pasinetti case). For tax rates above* τ_{lim}, *only equilibria with solely workers exist.*

(b) This relationship is monotone: the higher the value of γ, *the lower the tax rate at which capitalists vanish.*

[14]For the content of Propositions 3 and 4, the weaker claim $\rho_c + \delta_K > \alpha$ would suffice for Part (b) of the assumption. See Section 7 for calibration of the model, which includes employing time steps of 30 years.

Proof. It is straightforward to show that $\widetilde{S}/\widetilde{K}(\tau;\gamma)$ is monotonically increasing in τ and γ for $\tau, \gamma \in (0,1)$, keeping the other parameters fixed. Moreover, it can be established, using Equation (24) that

$$\lim_{\tau \to 1} \frac{\widetilde{S}}{\widetilde{K}}(\tau, \gamma) = \infty.$$

So as $\widetilde{S}/\widetilde{K}(\tau;\gamma)$ is continuous in $\tau \in (0,1)$, the proposition follows from the Intermediate Value Theorem.

□

Now consider the case $\gamma < 0$, that is, the substitution elasticity between capital and labor is smaller than 1.

Proposition 4. *Let production be specified as above. Assume $\gamma < 0$ and that assumption 1 still holds.*

(a) For every γ, there exists a capital tax rate, such that either capitalists vanish (Anti-Pasinetti case) or workers vanish (Anti-Anti-Pasinetti case).

(b) In both cases, the relationship is monotone: For the Anti-Pasinetti case, the higher the elasticity, the lower the tax rate at which capitalists vanish. For the Anti-Anti-Pasinetti case, the lower the elasticity, the lower the tax rate at which workers vanish.

The idea of the proof is to realize that for $\gamma < 0$ with $|\gamma|$ small, the function $\widetilde{S}/\widetilde{K}(\tau;\gamma)$ has a unique maximum that may or may not be greater than 1 depending on parameter choices.

Proof. See Appendix D.3. □

The following corollary characterizes exactly under which condition the Anti- and the Anti-Anti-Pasinetti case occur for $\gamma < 0$.

Corollary 5. *Let production be specified as above and assume $\gamma < 0$. Assumption 1 is still given.*

(a) There exists $\gamma_1 < 0$, such that: If $\gamma > \gamma_1$, for every γ, there exists a capital tax rate, such that capitalists vanish (Anti-Pasinetti case). If $\gamma < \gamma_1$, for every γ, there exists a tax rate such that workers vanish (Anti-Anti-Pasinetti case).

(b) In both cases, the relationship is monotone: For the Anti-Pasinetti case, the higher the elasticity, the lower the tax rate extinguishing capitalists. For the Anti-Anti-Pasinetti case, the lower the elasticity, the lower the tax that extinguishes workers.

Proof. See Appendix D.4. □

The first part of the corollary says that whenγ is very small, a high capital tax so reduces capital and the share of labor that *relatively* relatively disappears. The previous results follow from the fact that, given our specification of the production function, for *fixed* τ, the workers' wealth share increases in γ. This is a consequence of the fact that, so long as there are capitalists, the interest rate remains fixed by the capitalists' time preference rate even if the elasticity between capital and labor is changed. In our formulation, workers' fixed supply of labor is worth more the higher γ, so that they save more, crowding out capitalists" capital. This is not a general property, but is a consequence of the specification of our CES production function.[15]

6 Robustness: Public investment that enhances capital or generates rents

We analyze the robustness of the findings from Section 5 by considering alternative ways in which public investment might act on the economy: in particular, public capital can be an imperfect substitute for private capital, as in the case of state-owned companies. This can happen when public investment augments the interest rate (Section 6.1). Alternatively both labor- and capital-enhancing public investment can generate rents, implying that firms make profits (Section 6.2). The final subsection sketches a more general formulation with classes holding different capital goods and distinct forms of public investment (Subsection 6.3).

6.1 Capital-enhancing public investment

As an alternative to labor-enhancing public investment such as education, one can study capital-enhancing public investment. Core infrastructure investments may be plausibly represented as predominantly capital-, not labor-enhancing. In this subsection, we consider a variant of our model, introducing capital-enhancing public investment in a way entirely symmetric to Section 5. For this case, assume that there is a constant-returns-to-scale sub-production function H of both types of capital

$$H(K_t, P_t) = P_t H(K_t/P_t, 1). \tag{25}$$

[15]Reducing the elasticity of substitution simply means increasing the curvature. If this is done around the initial equilibrium point, the marginal rate of substitution (and hence the wage) remains unchanged at the point. The effect of a given change in the before tax return on capital on the capital-effective labor ratio depends, in turn, on the elasticity of substitution.

Total production is then given by $F(H, L)$ and is constant-returns to scale in H and L. With this modification, one defines the wage as:

$$w_t = \frac{\partial F(H_t, L)}{\partial L} \tag{12a}$$

Here we assume that public investment modifies the return to capital, so that profit maximization yields the following rate of return:

$$r_t + \delta_K = \frac{\partial F(H_t, L)}{\partial H_t}. \tag{13a}$$

Further, Equations (1) to (10) of the original model are assumed to hold, but with K, K^c and S replaced by H, H^c and H^S, that is H^c is capitalists' aggregate capital holding and H^S is workers aggregate capital.

Before proceeding, we note the following intuition: Even if now capital is augmented by public investment, not labor, the role of the elasticity between labor and (augmented) capital for studying limiting behaviour remains the same. Public investment that augments the productivity of capital means more effective capital supply, which would normally increase the share of labor so long as the elasticity of substitution is not too large (as opposed to too small, see earlier). However, it turns out that, for limiting behaviour as the capital tax is very large, the effect from taxing capital is decisive in our model *as long as the factor share accruing to (unaugmented) labor remains strictly positive.* It is unsurprising and true, though, that it is quantitatively more difficult to decrease wealth inequality when public investments augments capital, not labor. Only the qualitative result remains the same.

We now formally prove that the switch to augmenting capital, rather than labor, by public investment leaves the distributional outcome qualitatively unchanged *in the limit.* Assuming that a two-class steady steady exists, one can conclude that

$$\frac{\widetilde{H}^S}{\widetilde{H}}(\tau) = \frac{1}{2+\rho_w}\widetilde{\Omega}_L(\tau)/\widetilde{\Omega}_H(\tau)\Big(\frac{\rho_c}{1-\tau} + \delta_k\Big) \tag{26}$$

(with Ω_L and Ω_H the respective factor shares). It is again immediate that, since $\rho_c/(1-\tau)$ tends to infinity as $\tau \to 1$, if the factor share accruing to (unaugmented) labor is strictly positive as $\tau \to 1$, then $\frac{\widetilde{S}}{\widetilde{K}}(\tau)$ exceeds 1 (and in fact diverges). If for some tax rate both classes exist, then there will be a tax rate at which capitalists vanish, by the Intermediate Value Theorem. One can moreover establish the analog of Proposition 1, with $h(\tau) = H(\tau)/L$:

Proposition 6. *Assume* $0 < \frac{\widetilde{H_S}}{\widetilde{H}}(\epsilon) < 1$, *i.e. for small taxes rates* $\epsilon > 0$ *both classes coexist. Further assume* $\lim_{\tau\to 1} h(\tau) = 0$. *If* $\sigma(0) > 1$, *with* $\sigma(0)$ *the limiting elasticity of substitution as* $\tau \to 1$, *then there always exists a capital tax rate* $\tau_{\lim}$ *such that capitalists vanish (Anti-Pasinetti case).*

Proof. Equation (26) can be derived by analogy to the case of Equation (19), so the result follows by the proof of Proposition 1. □

Once again, there is a presumption that the capital tax will lead to lower wealth inequality, even if it finances capital-enhancing public investment. In Appendix E.1 we show additionally that for an explicit production function *in the limit* it is *qualitatively* irrelevant whether public investment is labour- or capital-enhancing.

In absolute terms, it of course matters which production factor is augmented by public investment. However, this variant of the model might not be convincing if one objects to the idea that public capital adds to the stock of private capital holdings. We next explore the alternative that public capital is a fully separate production factor that yields rents.

6.2 Public investment that creates rents appropriated by firms

Consider a version of the model in which public investment generates a return, which firms obtain as profits:

$$\Pi_t = \frac{\partial F(P_t, K_t, L)}{\partial P} P_t. \tag{27}$$

In Equation (2) we assumed that capitalists appropriate profits, for example as shareholders of the firms. Alternatively, one may think of the government as appropriating the rent and redistributing the returns to the capitalists. In this version, factor returns are given by

$$w_t = \frac{\partial F(K_t, L, P_t)}{\partial L}. \tag{12b}$$

and

$$r_t + \delta_K = \frac{\partial F(K_t, L, P_t)}{\partial K_t}. \tag{13b}$$

This formulation is plausible if the capitalist (who is also a shareholder of the firm) does not optimize for the rents he may receive as the government provides public investment. This would be so if there are many firms. If we think of the rents on public capital being appropriated in proportion to K, then an individual who invests more appropriates more of the public capital, so to him, the observed return to capital is the marginal return to private capital plus his increased share of the rents of public capital.

Findings from Section 5 are robust up to a multiplicative constant representing the productivity of public capital for the case of labor-enhancing public investment that generates rents. Let us consider the case in which public capital is labor-enhancing in the sense of Equation (11). From Equation (17), one immediately obtains a modified expression with factor shares Ω_J, Ω_K :

$$\frac{\widetilde{S}}{\widetilde{K}}(\tau) = \frac{1}{2+\rho_w} \frac{\partial \widetilde{J}}{\partial L} \widetilde{\Omega}_J(\tau)/\widetilde{\Omega}_K(\tau) \Big(\frac{\rho_c}{1-\tau} + \delta_k \Big). \tag{28}$$

In comparison with Equation (19), the marginal product of the composite factor with respect to labor enters as an additional multiplicative term.

Assume an explicit parametrization for J :

$$J_t = P_t^{\beta} L^{(1-\beta)}, \tag{29}$$

with $\alpha + \beta < 1$, to exclude the case of long-run or explosive growth. Then $\partial J/\partial L = (1-\beta)$. For a parametrized CES function as in Section 5, one thus finds that all results in Section 5 hold, but are modified by the multiplicative constant $(1-\beta)$.

Importantly, in this formulation with rents obtained as profits, one can treat analytically the case in which capitalists also receive labor income, in view of the evidence that those at the top of the wealth distribution also have significant labor income (Berman and Milanović, 2020; Ranaldi and Milanović, 2022). We find that, by comparison to a case in which only workers provide labor, they are relatively worse off, as expected.

Assume total labor is divided between workers' labor L_c and capitalists' labor L_w. Then

$$\widetilde{S}/\widetilde{K} = 1/(2+\rho_w)(\widetilde{w} L_w)/\widetilde{K}, \tag{30}$$

so by comparison to a case in which only workers provide labor, they are relatively worse off, as expected. However, by analogy to Proposition 1, one can still show that the elasticity of substitution between capital and labor determines whether there exists a tax rate at which capitalists disappear. So the results of this manuscript do not change qualitatively when it is assumed that capitalists also earn labor income.[16]

We study further details of this approach in Appendix E and note the following results here: For capital-enhancing public investment that generates rents, the Anti-Anti-Pasinetti case, workers disappearing, can still occur for poor substitution possibilities between aggregate capital and labor when the different forms of capital are highly substitutable (Appendix E.2). Finally,

[16] Berman and Milanović (2020) stress that in the current US economy high labor income is observed together with high wealth more than a few decades ago. A line of inquiry characterizing capital-tax financed public investment different from the one pursued in this contribution could therefore examine its effect on human capital transmission. Such an analysis could be especially relevant in societies in which the wealthy also receive high labor incomes.

for the special case of perfect substitutability between private and public capital, the tax rate at which a switch from the Pasinetti to the Anti-Pasinetti regime occurs is determined explicitly (Appendix E.3).

6.3 A more general formulation of public investment

There are two more possible criticisms of the above analysis. The first is that it ignores differences in the kinds of capital goods in which life-cycle savers and capitalists save. The former have less wealth and are accordingly naturally more risk averse. If there are costs associated with portfolio diversification and obtaining information concerning the relative merits of different assets, it is natural that (at any level of wealth holding) they are less diversified and that they spend less on information acquisition. Data bear out that there are large differences in compositions of assets and liabilities. These differences can have important implications for the distributive consequences of different policies: for instance, if wealth is understood as properly composed of productive capital holdings and fixed factor rents, taxing either component has different effects (Franks et al., 2018; Stiglitz, 2016b). Furthermore, since equities are disproportionately owned by the wealthy (capitalists, in our model), monetary policies like quantitative easing which disproportionately benefits equity owners contributed to an increase in wealth inequality (Galbraith, 2012; Stiglitz, 2010, 2015a; Turner, 2017).

Secondly, public investment in physical capital can take on a number of forms.[17] It can increase or decrease workers' wages, or increase or decrease the returns to private investment in capital. In the simple specifications explored so far, public capital is complementary to both labor and private capital, but that is not true more generally. (In a constant returns to scale production function with two factors, the two factors are necessarily complements, but in a production function with three or more factors, all that one can say is that each factor must be complementary with at least one other factor, i.e. $F_{ij} > 0$ for some j for every i.) Thus, we can formulate a more general production function

$$Y = F(L, K_w, K_c, P_1, P_2) \tag{31}$$

in which workers' and capitalists' capital may not be perfect substitutes for each other, and P_1 and P_2 are different forms of public goods, with, say

$$F_{L_{P_1}} > 0, F_{L_{P_2}} < 0, F_{K_w K_c} < 0 \tag{32}$$

and

$$F_{K_c P_1} < 0 \qquad \text{and} \qquad F_{K_c P_2} > 0. \tag{33}$$

[17]Empirical studies generally find an inequality-reducing effect of investment in infrastructure, for example, see Calderón and Chong (2004); Calderón and Servén (2014), but stress the heterogeneity of the impact of public investment on growth and inequality.

A tax on the return to particularly capitalists' capital with the proceeds invested in P_1 could (a) lead to an increase in wages and thereby K_w; but (b) leads to a decrease in capitalists' capital, so that wealth inequality would decrease. On the other hand, if the proceeds of the tax were invested in P_2, and $F_{K_c P_2}$ is large enough, then K_c might have to increase $F_{K_c P_2}$ to drive down the marginal return to private investment to the long-run equilibrium value, while wages and therefore K_w actually decrease. In this case, wealth inequality has increased. Government investment in research to create human-replacing robots is an example of public investment decreasing the return to labor, and government investment in roads may be an example of public investment which increased the private returns to a particular kind of private investment, railroads. In short, once one takes into account the variety of forms of public investment, it is clear that the incidence of a capital tax with proceeds invested depends on *which* investments are made.

7 Application: Quantitative implications of labor-enhancing public investment

In this section we calibrate the version of the model with labor-enhancing public investment (Section 5) to quantify the theoretical properties. We numerically determine a set of threshold values for the substitution elasticity between capital and labor ($\sigma = 1/(1-\gamma)$) and the capital tax rate (τ), as they completely determine which class benefits from increasing capital taxes to finance public investment. We numerically solve versions of Equation (24) and a set of equations determining the steady-state solution. The rationale for the simulation is to highlight that the threshold value for capital tax-financed public investment to decrease wealth inequality for all tax rates falls within the range of values for the substitution elasticity about which there is empirical disagreement (Chirinko, 2008; Piketty and Saez, 2014; Rognlie, 2014). For an extensive numerical treatment of a related model, covering a broader range of financing options and quantification outside of the steady-state, see Klenert et al. (2018).

The main results are illustrated in Figures 2 and 3. Figure 2 shows the wealth ratio changes with increasing capital taxes, for different elasticities. Figure 3 shows all conceivable model outcomes and the prevailing regimes in the form of a "phase diagram". To the left of the gray line, capitalists are better off in relative terms from an increase in capital taxes, to its right, workers are relatively better off.[18] Figure 4 translates this to *absolute terms*,

[18]Figure 3 is based on Equation (24). The Anti-Pasinetti frontier is given by setting the expression in Equation (24) to zero, while the Anti-Anti-Pasinetti frontier results from setting it to 1. The gray line in the center, determining which agent benefits from the

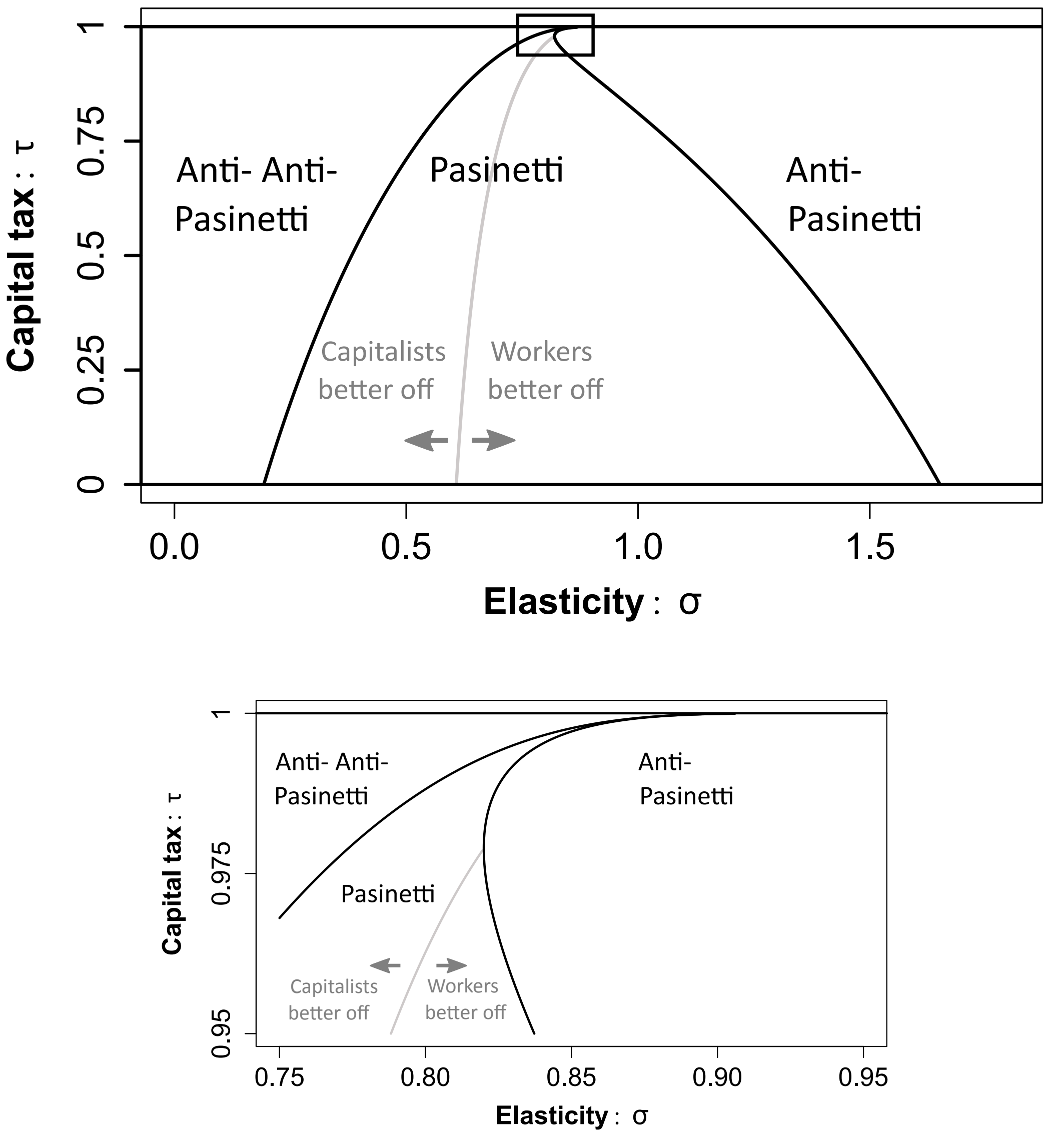


Figure 3: Equilibrium outcomes of the model with labor-enhancing public investment as a function of the elasticity of substitution between capital and labor σ and the capital tax rate τ. The lower half is an enlarged representation of the rectangle in the upper part. The gray line represents the frontier from which on capital tax-financed public investment harms or benefits either capitalists or workers *in relative terms*, i. e. have a higher share of capital. Cases: 'Pasinetti': both classes exist; 'Anti-Pasinetti': only workers exist; 'Anti-Anti-Pasinetti': only capitalists exist and no steady-state exists.

addressing the question when workers are actually better off in capital hold-

policy in relative terms, is computed by setting the first derivative of Equation (24) to zero.

ings (and hence welfare).[19] Here we find that the higher the elasticity, the higher can be the tax rate up to which workers are still better off in absolute terms.[20]

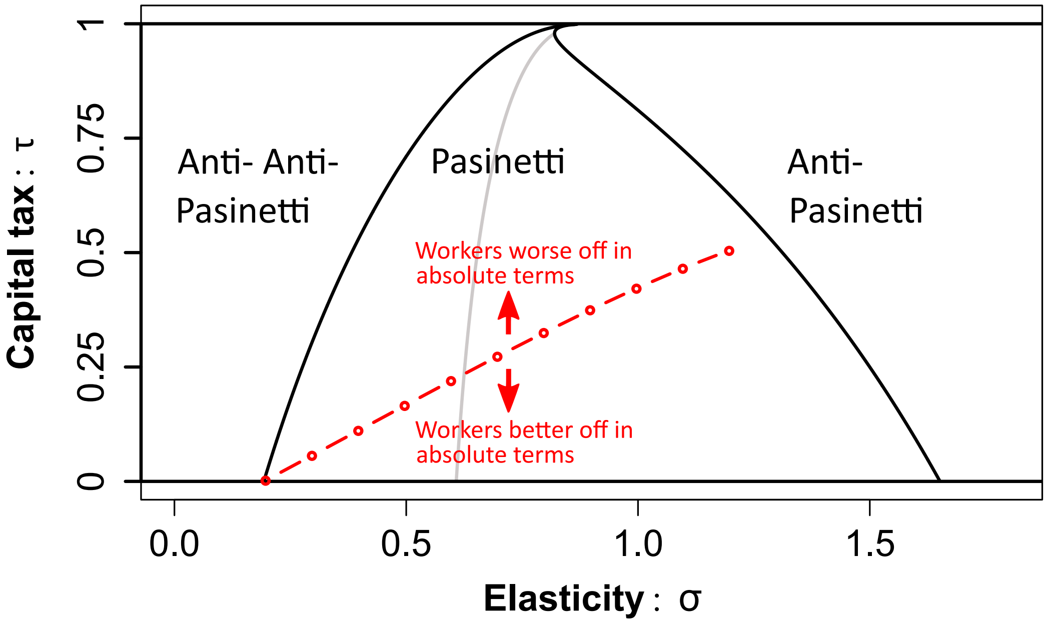


Figure 4: Equilibrium outcomes of the model with public investment as a function of the elasticity of substitution between capital and labor σ and the capital tax rate τ. In addition to Figure 3, this graph shows up to which tax rates workers are better off in *absolute terms*, i. e. have more savings (and utility), as they are benefitting from the effect of public investment on their wages. Cases: 'Pasinetti': both classes exist; 'Anti-Pasinetti': only workers exist; 'Anti-Anti-Pasinetti': only capitalists exist and no steady-state exists.

The calibration of the model, which is summarized in Table 1, is justified as follows: The capital share of income α is set at 0.38, in accordance with observations that in OECD countries the labor share of income was dropping

[19] We emphasize these are only steady state comparisons. If capitalists have more capital in the long run, then there must be capital accumulation, so in the intervening generations consumption must be less than it would otherwise have been. In the short run, the after-tax return is lowered, and capitalist' well-being is monotonic in after tax returns.

[20] The model is such that workers' consumption and welfare is also maximized at the maximal capital stock, see Mattauch et al. (2016) for an analytical solution of absolute improvements under restrictive conditions. We calculated the maximum capital stock of workers as a function of the tax rate numerically. For that purpose we use the algebraic modelling software GAMS (Rosenthal, 2014) to solve the system of nonlinear equations given by Equations (8a,9,12,13,15,16,23). For verification that the model converges to the steady state for almost the entire Pasinetti regime depicted in the figures of this section, see Appendix C.

from 66.1 % to 61.7 % from 1990 to 2009 (OECD, 2012). The productivity of public capital β, is chosen at 0.2 (Bom and Ligthart, 2014) in accordance with earlier estimates, which also suggest it is under-provided (Aschauer, 1989; Bom and Ligthart, 2014; Gramlich, 1994). As we focus on the case of labor-enhancing public investment as specified in Section 5, we parametrize the general sub-function $J(L, P)$ assumed in that section with $J_t = P_t^{\beta} L^{(1-\beta)}$, with $0 < \beta < 1$, and $\alpha + \beta < 1$, so that there is no long-run or explosive growth. Time is measured in steps of 30 years because workers are assumed to live for two periods (and capitalists are dynasties perfectly altruistic to their descendants); we choose corresponding values for time preference and depreciation rates.

Because of the absence of reliable data for calibrating the wealth distribution for the OECD (Alvaredo et al., 2018), we calibrate the model to the U.S. wealth distribution (Wolff, 2010) and check robustness below. In 2007, 62 % of net worth were held by the top 5 % of the population and almost 38 % of net worth by the remaining 95 %. We thus set the time preference rates ρ_c and ρ_w such that for a capital tax of 21 %, (the average capital tax rate in OECD countries between the years 1970 and 2000, Carey and Rabesona, 2002) and an elasticity of substitution between capital and labor of 1, this wealth distribution results.[21] In accordance with evidence that richer households are more patient (Lawrance, 1991; Green et al., 1996; Epper et al., 2020), time preference rates of capitalists are chosen significantly lower than that of workers.

Parameter	Standard value	Corresponding annual value
ρ_c	0.56	1.5%
ρ_w	3.98	5.5%
δ_k	0.7	4%
δ_P	0.7	4%
α	0.38	–
β	0.2	–
L	100	–

Table 1: Model calibration

Finally, we conduct a sensitivity analysis of the threshold value of the elasticity σ_1. For elasticities greater than this value, high tax rates lead to an Anti-Pasinetti regime (capitalists disappearing). The threshold is given by

[21]See Section 6.2 and Appendix E for the qualitatively similar case that capitalists also have labor income. Also Chancel et al. (2022) confirms that for selected OECD countries and the top 10 % and bottom 50 % wealth share, the wealth distribution is remarkably stable over the past decade.

setting Equation (D.18) to 1 and thus depends only on the parameters ρ_w and α. Table 2 shows that the threshold is hardly dependent on economically plausible values for workers' time preference rate for the capital share of income. The latter is relevant as our model does not distinguish between (direct) capital holdings of households and capital holdings by firms, which could arguably influence the appropriate choice for the capital share (see Flores (2021)).

ρ_w	σ_1	α	σ_1
3	0.812	0.33	0.793
3.2	0.814	0.34	0.799
3.4	0.816	0.35	0.804
3.6	0.817	0.36	0.81
3.8	0.819	0.37	0.815
3.98	**0.82**	**0.38**	**0.82**
4.2	0.822	0.39	0.825
4.4	0.823	0.4	0.83
4.6	0.824	0.41	0.835
4.8	0.825	0.42	0.839
5	0.826	0.43	0.844

Table 2: Dependency of the threshold elasticity on capital share and workers' time preference rate. Standard values in bold.

8 Conclusion

This paper examines whether taxing capital at higher rates in order to finance underfunded public capital helps to mitigate wealth inequality. Wealth inequality continues to rise in rich countries, which is at least partially due to heterogeneous saving behavior across households. We consider disparities in saving behavior in a way that reflects a fundamental distinction in most advanced countries: Rich individuals have a higher savings rate (lower time preference rates), obtain a greater share of their income from capital and save for posterity, not for retirement, when compared to the rest of society. Our study develops the simplest possible framework representing these disparities by combining a standard life-cycle saving working class with dynastically saving top earners. We analyze this framework in a general equilibrium setting with a neoclassical production function incorporating public capital as a separate production factor, and show it is a consistent mode of analysis for representing households with different saving behaviour.

We prove that capital taxation is successful in reducing wealth inequality if the expenditure is used for public investment under mild conditions likely to be satisfied. As expected, the result depends on the substitutability of capital and labor. Our results can be seen as confirming that capital tax-financed public investment—the major policy recommendation resulting from Piketty (2014)—reduces inequality in wealth in the long term, however, they can also be seen as a note of caution against this policy recommendation if the elasticity of substitution between capital and labor is sufficiently low. An important contribution of our paper is to estimate, using a calibrated CES model, the critical values of the tax rate below which capital taxation with proceeds spent on public investment in human capital leads to a reduction in inequality. We find that for standard parameter values the critical threshold between the different limiting cases is around 0.82. This simulated outcome is very robust to changes in the specification of relevant parameters. While empirically there is disagreement about the value of the substitution elasticity (Chirinko, 2008; Karabarbounis and Neiman, 2014; Piketty and Saez, 2014; Knoblach et al., 2016), for plausible estimates of the elasticity, capital taxation leads to reduced inequality; still, it is worth noting that for some of the lower elasticities noted in the literature, capital taxation can decrease equality only to a very limited extent, and for some, it may even increase inequality.

Our model has the feature that if the elasticity of substitution is high enough (but still less than one), a high enough tax squeezes out capitalists—only life cycle savers remain.[22] Taking into account the benefits of public investment is critical to the distributional analysis: When the expenditure effect is separated, as is conventional, we confirm capital taxation alone does not reduce inequality. While we show that so long as the returns to investing in human capital are large enough, a small capital tax *always* improves workers' wages, we demonstrate numerically that even for lower elasticities, wealth inequality decreases and workers are better off in absolute terms, so long as capital taxes are only moderately high.

We do not provide a full welfare analysis here, but point out that steady state social welfare is higher with capital taxation with proceeds spent on public investment, so long as the social welfare function is sufficiently equalitarian, so long as the initial differences in steady state per capita incomes between capitalists and workers are large enough, and so long as the elasticity of substitution is not too small. To pin down specific welfare-optimal policies (for instance, the optimal level of capital taxation), not merely Pareto-

[22]It is obvious as one approaches this limit, wealth equality increases, with an increasing fraction of wealth that of workers. For sake of completeness, we also noted that for low enough values of the elasticity of substitution between capital and labor, there exists a positive capital tax rate at which life-cycle savers disappear. However, our numerical results suggest this is a case with little practical relevance.

improvements, that a rigorous development of a social welfare function for the framework we are studying would need to extend the Calvo and Obstfeld (1988) time-consistent social welfare function in an overlapping generations model (see also Franks and Edenhofer (2020)).

In most advanced countries for several decades there has been a secular trend of increased inequality. There is growing recognition that this may have political as well as economic consequences, and that the full adverse economic consequences may not be captured in standard models such as those presented here. Artificial intelligence introduces further important complexities: it may change both the elasticities of substitution and the returns to human capital, two of the key determinants of the potential effectiveness of capital taxation in reducing inequality. More recently, there is growing concern that Artificial intelligence will therefore exacerbate these trends, making it all the more important a better understanding of the instruments by which inequality might be addressed.

Acknowledgements We thank Max Franks, Cameron Hepburn, Olga Heismann, Katharina Jochemko, Ulrike Kornek, Marc Morgan, Salvatore Morelli, Jacquelyn Pless and Anselm Schultes for helpful comments and Fiona Spuler and Simona Sulikova for excellent research assistance. We further thank seminar audiences in Berlin and Oxford, at ZEW as well as participants of meetings of the European Economic Association, the Journées Louis-André Gérard Varet, the Royal Economic Society, the International Economic Association, the Society for the Study of Economic Inequality and the Verein fuer Socialpolitik and for useful comments. Linus Mattauch's research was supported by a postdoctoral fellowship of the German Academic Exchange Service (DAAD). The article is also part of a project that has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 681228. The content of this article does not reflect the official opinion of the European Commission; responsibility for the information and views expressed in the article lies entirely with the authors. Declarations of interest: none.

References

Aiyagari, S. R. (1994). Uninsured idiosyncratic risk and aggregate saving. *The Quarterly Journal of Economics*, 109(3):659–684.

Alvaredo, F., Chancel, L., Piketty, T., Saez, E., and Zucman, G. (2017). Global inequality dynamics: New findings from wid. world. *American Economic Review*, 107(5):404–09.

Alvaredo, F., Chancel, L., Piketty, T., Saez, E., and Zucman, G. (2018). WID – World Wealth and Income Database. wid.world, retrieved 2018-02-07.

Aschauer, D. A. (1989). Is public expenditure productive? *Journal of Monetary Economics*, 23(2):177–200.

Atkinson, A. B. and Stiglitz, J. E. (1969). A new view of technological change. *The Economic Journal*, 79(315):573–578.

Atkinson, A. B. and Stiglitz, J. E. (1976). The design of tax structure: direct versus indirect taxation. *Journal of Public Economics*, 6(1-2):55–75.

Attanasio, O. P. (1994). Personal saving in the United States. In Poterba, J. M., editor, *International Comparisons of Household Saving*, pages 57–124. The University of Chicago Press, Chicago.

Baranzini, M. (1991). *A theory of wealth distribution and accumulation.* Oxford University Press.

Baranzini, M. L. and Mirante, A. (2021). Pasinetti's theorem: A narrow escape, for what was to become an inexhaustible research programme. *Structural Change and Economic Dynamics*, 59:470–481.

Barelli, P. and de Abreu Pessôa, S. (2003). Inada conditions imply that production function must be asymptotically Cobb–Douglas. *Economics Letters*, 81(3):361–363.

Becker, R. A. (1980). On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households. *The Quarterly Journal of Economics*, 95(2):375–382.

Berman, Y. and Milanović, B. (2020). Homoploutia: Top labor and capital incomes in the united states, 1950-2020. Stone Center Working Paper Series N. 28.

Bevan, D. L. (1974). Savings, inheritance and economic growth in the presence of earnings inequality. Oxford University mimeo.

Bevan, D. L. (1979). Inheritance and the distribution of wealth. *Economica*, 46(184):381–402.

Bevan, D. L. and Stiglitz, J. E. (1979). Intergenerational transfers and inequality. *The Greek Economic Review*, 1(1):8–26.

Bewley, T. (1977). The permanent income hypothesis: A theoretical formulation. *Journal of Economic Theory*, 16(2):252–292.

Bom, P. R. D. and Ligthart, J. E. (2014). What have we learned from three decades of research on the productivity of public capital? *Journal of Economic Surveys*, 28(5):889–961.

Calderón, C. and Chong, A. (2004). Volume and quality of infrastructure and the distribution of income: an empirical investigation. *Review of Income and Wealth*, 50(1):87–106.

Calderón, C. and Servén, L. (2014). Infrastructure and growth. In Durlauf, S. N. and Blume, L. E., editors, *The New Palgrave Dictionary of Economics. Online Edition.* Palgrave Macmillan, Basingstoke. Accessed: 1 December 2014.

Calvo, G. A. and Obstfeld, M. (1988). Optimal time-consistent fiscal policy with finite lifetimes. *Econometrica*, 56(2):411–432.

Carey, D. and Rabesona, J. (2002). Tax ratios on labour and capital income and on consumption. *OECD Economic Studies*, 35(2):129–174.

Champernowne, D. G. (1953). A model of income distribution. *The Economic Journal*, 63(250):318–351.

Chancel, L., Piketty, T., Saez, E., and Zucman, G. (2022). World Inequality Report 2022. World Inequality Lab, 2021.

Chatterjee, S. and Turnovsky, S. J. (2012). Infrastructure and inequality. *European Economic Review*, 56(8):1730–1745.

Chirinko, R. S. (2008). σ: The long and short of it. *Journal of Macroeconomics*, 30(2):671–686.

Diamond, P. and Saez, E. (2011). The case for a progressive tax: from basic research to policy recommendations. *The Journal of Economic Perspectives*, 25(4):165–190.

Diaz-Gimenez, J., Glover, A., and Rios-Rull, J.-V. (2011). Facts on the Distributions of Earnings, Income, and Wealth in the United States: 2007 Update. *Federal Reserve Bank of Minneapolis Quarterly Review*, 34(1).

Dynan, K. E., Skinner, J., and Zeldes, S. P. (2004). Do the rich save more? *Journal of Political Economy*, 112(2):397–444.

Edenhofer, O., Mattauch, L., and Siegmeier, J. (2015). Hypergeorgism: When rent taxation is socially optimal. *FinanzArchiv/Public Finance Analysis*, pages 474–505.

Epper, T., Fehr, E., Fehr-Duda, H., Kreiner, C. T., Lassen, D. D., Leth-Petersen, S., and Rasmussen, G. N. (2020). Time discounting and wealth inequality. *American Economic Review*, 110(4):1177–1205.

Flores, I. (2021). The capital share and income inequality: Increasing gaps between micro and macro-data. *The Journal of Economic Inequality*, 19(4):685–706.

Foley, D. K. and Michl, T. R. (1999). *Growth and distribution.* Harvard University Press.

Franks, M. and Edenhofer, O. (2020). Optimal redistributive wealth taxation when wealth is more than just capital. CESifo Working Paper 8093.

Franks, M., Klenert, D., Schultes, A., Edenhofer, O., and Lessmann, K. (2018). Is capital back? the role of land ownership and savings behavior. *International Tax and Public Finance*, 25(5):1252–1276.

Gabaix, X. (2009). Power laws in economics and finance. *Annual Review of Economics*, 1(1):255–294.

Galbraith, J. K. (2012). *Inequality and instability: A study of the world economy just before the great crisis*. Oxford University Press.

Gramlich, E. M. (1994). Infrastructure investment: a review essay. *Journal of Economic Literature*, 32(3):1176–1196.

Green, L., Myerson, J., Lichtman, D., Rosen, S., and Fry, A. (1996). Temporal Discounting in Choice Between Delayed Rewards : The Role of Age and Income. *Psychology and Aging*, I:79–84.

Hanushek, E. A. and Woessmann, L. (2008). The role of cognitive skills in economic development. *Journal of economic literature*, 46(3):607–68.

Heijdra, B. J. (2009). *Foundations of Modern Macroeconomics*. Oxford University Press, Oxford.

Jones, C. I. (2015). Pareto and Piketty: The macroeconomics of top income and wealth inequality. *The Journal of Economic Perspectives*, 29(1):29–46.

Judd, K. L. (1985). Redistributive taxation in a simple perfect foresight model. *Journal of Public Economics*, 28(1):59–83.

Karabarbounis, L. and Neiman, B. (2014). The global decline of the labor share. *The Quarterly Journal of Economics*, 129(1):61–103.

Klenert, D., Mattauch, L., Edenhofer, O., and Lessmann, K. (2018). Infrastructure and inequality: Insights from incorporating key economic facts about household heterogeneity. *Macroeconomic Dynamics*, 22:864–895.

Knoblach, M., Rößler, M., and Zwerschke, P. (2016). The elasticity of factor substitution between capital and labor in the US economy: a meta-regression analysis. CEPIE Working Paper 03/16.

Korinek, A. and Stiglitz, J. E. (2017). Artificial intelligence and its implications for income distribution and unemployment. NBER Working Paper 24174.

Krusell, P. and Smith, Jr, A. A. (1998). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5):867–896.

Lawrance, E. C. (1991). Poverty and the rate of time preference: evidence from panel data. *Journal of Political Economy*, 99(1):54–77.

Mattauch, L., Edenhofer, O., Klenert, D., and Bénard, S. (2016). Distributional Effects of Public Investment when Wealth and Classes are Back. *Metroeconomica*, 67(3):603–629.

Mattauch, L., Siegmeier, J., Edenhofer, O., and Creutzig, F. (2018). Financing public capital when rents are back: a macroeconomic henry george theorem. *FinanzArchiv/Public Finance Analysis*, 74(3).

Meade, J. E. (1966). The outcome of the pasinetti-process: a note. *The Economic Journal*, 76(301):161–165.

Michl, T. R. (2009). *Capitalists, workers, and fiscal policy: a classical model of growth and distribution.* Harvard University Press, Cambridge, MA.

Novokmet, F., Piketty, T., and Zucman, G. (2017). From Soviets to Oligarchs: inequality and property in Russia, 1905-2016. NBER Working Paper 23712.

OECD (2012). *Labour Losing to Capital: What Explains the Declining Labour Share?*, chapter 3, pages 109–161. OECD.

Pasinetti, L. L. (1962). Rate of profit and income distribution in relation to the rate of economic growth. *Review of Economic Studies*, 29(4):267–279.

Piketty, T. (2014). *Capital in the Twenty-First Century.* Harvard University Press.

Piketty, T. and Saez, E. (2013). A theory of optimal inheritance taxation. *Econometrica*, 81(5):1851–1886.

Piketty, T. and Saez, E. (2014). Inequality in the long run. *Science*, 344(6186):838–843.

Piketty, T. and Zucman, G. (2014). Capital is back: Wealth-income ratios in rich countries 1700-2010. *Quarterly Journal of Economics*, 129(3):1255–1310.

Piketty, T. and Zucman, G. (2015). Wealth and inheritance in the long run. In Atkinson, A. B. and Bourguignon, F., editors, *Handbook of income distribution. Vol. 2*, pages 1303–1368. Elsevier.

Ranaldi, M. and Milanović, B. (2022). Capitalist systems and income inequality. *Journal of Comparative Economics*, 50(1):20–32.

Rognlie, M. (2014). A note on Piketty and diminishing returns to capital. http://gabriel-zucman.eu/files/teaching/Rognlie14.pdf. retrieved 15-10-07.

Rosenthal, R. E. (2014). GAMS A User's Guide.

Saez, E. and Zucman, G. (2016). Wealth Inequality in the United States Since 1913: Evidence from Capitalized Income Tax Data. *Quarterly Journal of Economics*, 131(2):519–578.

Saez, E. and Zucman, G. (2020). The rise of income and wealth inequality in america: Evidence from distributional macroeconomic accounts. *Journal of Economic Perspectives*, 34(4):3–26.

Samuelson, P. A. and Modigliani, F. (1966). The Pasinetti paradox in neoclassical and more general models. *Review of Economic Studies*, 33(4):269–301.

Stiglitz, J. E. (1966). The distribution of income and wealth among individuals. Presented at the December 1966 meetings of the Econometric Society, San Francisco, December 29 (MIT mimeo).

Stiglitz, J. E. (1967). A Two-Sector Two Class Model of Economic Growth. *The Review of Economic Studies*, 34(2):227–238.

Stiglitz, J. E. (1969). Distribution of Income and Wealth Among Individuals. *Econometrica*, 37(3):382–397.

Stiglitz, J. E. (1978). Notes on estate taxes, redistribution, and the concept of balanced growth path incidence. *Journal of Political Economy*, 86(2, Part 2):S137–S150.

Stiglitz, J. E. (2010). *Freefall: America, free markets, and the sinking of the world economy*. WW Norton & Company.

Stiglitz, J. E. (2012). *The price of inequality: How today's divided society endangers our future*. WW Norton & Company.

Stiglitz, J. E. (2015a). Fed policy, inequality, and equality of opportunity. Roosevelt Institute Report.

Stiglitz, J. E. (2015b). New theoretical perspectives on the distribution of income and wealth among individuals: Part i. the wealth residual.

Stiglitz, J. E. (2015c). New theoretical perspectives on the distribution of income and wealth among individuals: Part ii: Equilibrium wealth distributions. NBER Working Paper 21190.

Stiglitz, J. E. (2015d). New theoretical perspectives on the distribution of income and wealth among individuals: Part iii: Life cycle savings vs. inherited savings. NBER Working Paper 21191.

Stiglitz, J. E. (2015e). New theoretical perspectives on the distribution of income and wealth among individuals: Part IV: Land and credit. National Bureau of Economic Research working paper 21192.

Stiglitz, J. E. (2016a). How to restore equitable and sustainable economic growth in the united states. *American Economic Review*, 106(5):43–47.

Stiglitz, J. E. (2016b). New Theoretical Perspectives on the Distribution of Income and Wealth Among Individuals. In Basu, K. and Stiglitz, J. E., editors, *Inequality and Growth: Patterns and Policy, Volume I: Concepts and Analysis.* IEA Conference Volume No. 156-I, Houndmills, UK and New York: Palgrave Macmillan.

Stiglitz, J. E. (2018a). Pareto efficient taxation and expenditures: Pre-and redistribution. *Journal of Public Economics*, 62:101–119.

Stiglitz, J. E. (2018b). Where modern macroeconomics went wrong. *Oxford Review of Economic Policy*, 34(1-2):70–106.

Stiglitz, J. E. (2021). Economic fluctuations and pseudo-wealth. National Bureau of Economic Research Working Paper w28415.

Straub, L. and Werning, I. (2020). Positive long-run capital taxation: Chamley-judd revisited. *American Economic Review*, 110(1):86–119.

Taylor, L. (2014). The triumph of the rentier? Thomas Piketty vs. Luigi Pasinetti and John Maynard Keynes. *International Journal of Political Economy*, 43(3):4–17.

Turner, A. (2017). *Between debt and the devil: money, credit, and fixing global finance.* Princeton University Press.

Wolff, E. (2017). Household wealth trends in the united states, 1962 to 2016: Has middle class wealth recovered? NBER Working Paper 24085.

Wolff, E. N. (1998). Recent trends in the size distribution of household wealth. *Journal of Economic Perspectives*, 12(3):131–150.

Wolff, E. N. (2010). Recent trends in household wealth in the United States: rising debt and the middle-class squeeze - an update to 2007. *The Levy Economics Institute Working Paper Collection 589.*

Wolff, E. N. and Zacharias, A. (2013). Class structure and economic inequality. *Cambridge Journal of Economics*, 37:1381–1406.

Zamparelli, L. (2017). Wealth distribution, elasticity of substitution and Piketty: an 'anti-dual' pasinetti economy. *Metroeconomica*, 68(4):927–946.

For online publication

Appendices

A Burden shifting in two-class models

In this manuscript we focus on cases in which the government uses the capital tax revenue for public investment, since this policy, under certain conditions, constitutes a Pareto improvement, i. e. it makes all classes better off in absolute terms (including in *welfare*), while a lump-sum return of capital tax proceeds does not (Mattauch et al., 2016; Klenert et al., 2018; Stiglitz, 2018a). If the capital tax revenue was instead used to finance lump-sum transfers to the workers, the burden of the capital tax would be fully shifted to the workers (see e. g. Stiglitz, 2016b). This would make workers worse off than before in absolute terms (in both savings and welfare), as shown by Stiglitz (2016b) and Stiglitz (2018a) for small tax rates. A capital tax in a two-class model hence only redistributes without making one class worse off if its proceeds are invested in public capital.

In this Appendix, we translate the formal arguments on burden shifting for this result to the model used in this manuscript (with endogenous savings) for completeness. We briefly sketch the proof of full burden shifting under a capital tax recycled as a lump-sum transfer and then outline why it is not conclusive for public investment.

Proposition 7. *If the capital tax revenue is redistributed lump-sum to the workers, the burden of the capital tax is shifted to the workers.*

In the following proof we work with per capita variables only. Let $k_t = K_t/L$, $f(k_t) = F(K_t, L)/L$ etc. as usual.

Proof. If the capital tax revenue were redistributed to the workers through a lump-sum transfer λ_t, the workers young-period budget equation would be given by:

$$w_t + \lambda_t = s_t + c_t^y. \tag{A.1}$$

The lump-sum transfer would be given by

$$\lambda_t = \tau r_t k_t. \tag{A.2}$$

The workers' per capita saving would be given by

$$s_t = \frac{1}{2+\rho_w}(w_t + \lambda_t). \tag{A.3}$$

In the following we show that $(\partial\tilde{s})/(\partial\tau) < 0$. We use Equation (A.2) and the fact that without public investment and in per capita variables $w_t = f(k_t) - f_k(k_t)k_t$. We use $f_k(k)$ as a shorthand for the first derivative of $f(k)$ with respect to k. Also, we only consider the change in steady-state values, so the steady-state interest rate is still determined by Equation (15).

$$\begin{aligned}
\frac{\partial\tilde{s}}{\partial\tau} &= \frac{\partial}{\partial\tau}\left(\frac{1}{2+\rho_w}\left(\left(f(\tilde{k}) - f_k(\tilde{k})k\right) + \tau r\tilde{k}\right)\right) \\
&= \frac{1}{2+\rho_w}\Big(f_k(\tilde{k})\frac{\partial\tilde{k}}{\partial\tau} - f_k(\tilde{k})\frac{\partial\tilde{k}}{\partial\tau} - f_{kk}(\tilde{k})\tilde{k}\frac{\partial\tilde{k}}{\partial\tau} \\
&+ \tilde{k}(f_k(\tilde{k}) - \delta_k) + \tau f_{kk}(\tilde{k})\tilde{k}\frac{\partial\tilde{k}}{\partial\tau} + \tau\frac{\partial\tilde{k}}{\partial\tau}(f_k(\tilde{k}) - \delta_k)\Big) \\
&= \frac{1}{2+\rho_w}\left(\left(-f_{kk}(\tilde{k})\tilde{k}(1-\tau) + \tau(f_k(\tilde{k}) - \delta_k)\right)\frac{\partial\tilde{k}}{\partial\tau} + (f_k(\tilde{k}) - \delta_k)\tilde{k}\right)
\end{aligned} \tag{A.4}$$

Applying the Implicit Function Theorem to Equation (15) yields: $\rho_c/(1-\tau)^2 1/f_{kk}(\tilde{k}) = \partial\tilde{k}/\partial\tau$. Hence, Equation (A.4) becomes:

$$\begin{aligned}
\frac{\partial\tilde{s}}{\partial\tau} &= \frac{1}{2+\rho_w}\Big(-\frac{\rho_c\tilde{k}}{(1-\tau)} + \frac{\rho_c\tau}{(1-\tau)^2}\frac{(f_k(\tilde{k}) - \delta_k)}{f_{kk}(\tilde{k})} + (f_k(\tilde{k}) - \delta_k)\tilde{k}\Big) \\
&= \frac{1}{2+\rho_w}\Big(\frac{\rho_c^2\tau}{(1-\tau)^3}\frac{1}{f_{kk}(\tilde{k})}\Big) < 0, \qquad \text{since} \qquad f_{kk}(\tilde{k}) < 0.
\end{aligned} \tag{A.5}$$

The last equality follows because the first and third summand are equal, applying several times Equations (13) and (15).

□

The last expression in this derivation is analogous to Equation (1.11) in (Stiglitz, 2015d).

Furthermore, note that if the capital tax revenue is invested in public capital, it is generally unclear if full burden shifting will occur, since the sign of $(\partial\tilde{S})/(\partial\tau)$ is ambiguous.[23] A non-technical way of thinking about this case would be to argue that public investment P follows a "Laffer curve". Therefore sign in Equation (A.7) below will be ambiguous. If the effect of a tax increase on augmented labor-income is very high, as can be expected for low tax rates, workers likely benefit from public investment in absolute terms. This corroborates the messages of Figure 4 in the main part.

To see this formally, note that for the case of labor-enhancing public investment, the change in workers' savings in the intensive form as in the proof

[23]This statement is in line with Footnote 32 in Stiglitz (2015c), although for a differing production structure.

of Proposition 7 is no longer a meaningful indicator of the distributional impact of the policy. We hence have to look at the change in workers' aggregate savings $(\partial\tilde{S})/(\partial\tau)$.

Aggregate savings are given by $S_t = \frac{1}{2+\rho_w} J_t w_t$. So the change in workers' aggregate savings induced by capital tax-financed public investment in the steady state is given by:

$$\begin{aligned}
\frac{\partial(\tilde{S})}{\partial\tau} &= \frac{1}{2+\rho_w}\frac{\partial}{\partial\tau}\left(\tilde{J}\left(f(\tilde{k}) - f_k(\tilde{k})k\right)\right)\\
&= \frac{1}{2+\rho_w}\left(\left(\frac{\partial\tilde{J}}{\partial\tau}\right)\left(f(\tilde{k}) - f_k(\tilde{k})k\right) + \tilde{J}\frac{\partial}{\partial\tau}\left(f(\tilde{k}) - f_k(\tilde{k})k\right)\right)\\
&= \frac{1}{2+\rho_w}\left(\left(\frac{\partial\tilde{J}}{\partial\tau}\right)\left(f(\tilde{k}) - f_k(\tilde{k})k\right) - \tilde{J}f_{kk}(\tilde{k})\tilde{k}\frac{\partial\tilde{k}}{\partial\tau}\right)\\
&= \frac{1}{2+\rho_w}\left(\left(\frac{\partial\tilde{J}}{\partial\tau}\right)\left(f(\tilde{k}) - f_k(\tilde{k})k\right) - \tilde{J}\tilde{k}\frac{\rho_c}{(1-\tau)^2}\right).
\end{aligned} \tag{A.6}$$

The second summand within the brackets is unambiguously negative, but the first summand is positive in case $\left(\frac{\partial\tilde{J}}{\partial\tau}\right)$ is, in which case the sign may or may not be positive. It thus remains to determine the sign of $\left(\frac{\partial\tilde{J}}{\partial\tau}\right)$:

$$\left(\frac{\partial\tilde{J}}{\partial\tau}\right) = \left(\frac{\partial}{\partial\tau}\right)\tilde{P}^{\beta}L^{(1-\beta)}. \tag{A.7}$$

Steady-state public investment levels are given by $\delta_P\tilde{P} = \tau\tilde{r}\tilde{K}$, with $\tilde{r} = \rho_c/(1-\tau)$, so the above equation becomes:

$$\begin{aligned}
\left(\frac{\partial\tilde{J}}{\partial\tau}\right) &= \left(\frac{\partial}{\partial\tau}\right)\left(\frac{\tau}{(1-\tau)}\frac{\rho_c}{\delta_P}\tilde{K}\right)^{\beta}L^{(1-\beta)}\\
&= L^{(1-\beta)}\left(\frac{\rho_c}{\delta_P}\right)^{\beta}\beta\left(\frac{\tau}{(1-\tau)}\tilde{K}\right)^{\beta-1}\left(\frac{\tilde{K}}{(1-\tau)^2} + \frac{\tau}{(1-\tau)}\frac{\partial\tilde{K}}{\partial\tau}\right).
\end{aligned} \tag{A.8}$$

Using the implicit function theorem (for the non-intensive version of the production function) on Equation (15) yields:

$$\frac{\partial\tilde{K}}{\partial\tau} = \left[\frac{\rho_c}{(1-\tau)} - F_{KP}\frac{\rho_c}{\delta_P(1-\tau)^2}\tilde{K}\right]\left(F_{KK} + F_{KP}\frac{\tau\rho_c}{(\delta_P(1-\tau))}\right)^{(-1)}. \tag{A.9}$$

The expression in Equation (A.9) can be bigger, smaller or equal to zero. Therefore the sign of $\frac{\partial(\tilde{J})}{\partial\tau}$ and hence $\frac{\partial(\tilde{S})}{\partial\tau}$ is ambiguous.

B Further results with general production function

B.1 Sufficient condition for workers to be absolutely better off

Consider the case in which tax revenues are allocated to human capital, the production function takes the form, using the constant-returns-to-scale properties and our normalization of $L = 1$,

$$F = j(\widetilde{K}\frac{\tau\rho_c}{(1-\tau)\delta_P})f(\kappa(K)) \tag{B.1}$$

where $\widetilde{\kappa}$ is the ratio of capital to "effective" labor, i.e.

$$\kappa \equiv \widetilde{K}/\frac{\tau\rho_c}{(1-\tau)\delta_P}. \tag{B.2}$$

Using standard results, it follows from Equation (13) that

$$r + \delta_K = f'(\widetilde{\kappa}), \tag{B.3}$$

so that, from Equation (15), for a given τ, there is a unique equilibrium value of $\widetilde{\kappa}$. From Equation (B.2), it follows that

$$\mathrm{d}\ln\widetilde{\kappa}/\mathrm{d}\ln\widetilde{K} = 1 - \eta \tag{B.4}$$

where

$$\eta \equiv \mathrm{d}\ln j/\mathrm{d}\ln\widetilde{P}, \tag{B.5}$$

the elasticity of effective labor supply with respect to public investment. We assume, for now, that (over the relevant range) public investment is highly productive (this corresponds to the assumption of underinvestment in the initial equilibrium), so $\eta > 1$. Thus, corresponding to $\widetilde{\kappa}$, there is a unique steady state value of $\widetilde{K}$. Note that an increase in the rate of capital taxation *increases* the overall level of capital—public investment so increases the productivity of labor that the return to capital is increased, and capitalists are induced to save more. Eventually, of course, the productivity of public capital may fall, so that $\eta < 1$, in which case the aggregate capital stock would fall.[24] But increasing the tax rate beyond that level would be counterproductive, because that would also lead to a lowering of public capital.

[24]If η is initially above unity, but then falls below unity, there could be two values of $\widetilde{K}$ corresponding to any value of $\widetilde{\kappa}$, i. e. consistent with the steady state equilibrium. At the higher value of $\widetilde{K}$, workers are better off (because j is higher). Provided η is not too high, capitalists are also better off (see discussion below.)

It also follows that we can solve uniquely for wages per efficiency unit (using Equation (12)):

$$\widetilde{w} = f(\kappa) - f'(\kappa)\kappa, \tag{B.6}$$

and thus enhanced wages per worker

$$\widetilde{W} = \widetilde{w}j = j\big(\frac{\rho_c \tau K(\kappa)}{(1-\tau)\delta_P}\big)[f(\kappa) - f'(\kappa)\kappa] \tag{B.7}$$

It then follows from the steady-state value of the interest rate that:

$$\mathrm{d}\widetilde{\kappa}/\mathrm{d}\tau = \frac{r}{(1-\tau)f''(\widetilde{\kappa})} < 0, \tag{B.8}$$

i. e. an increase in the capital tax has to lower the capital effective labor-ratio in order to increase the before tax return to capital. Thus from Equation (B.4), at least for small increase in the tax rate ($\eta > 1$), K increases, and that means j increases. It follows that we can determine the effect of wages on a change in τ :

$$\frac{\mathrm{d}\ln\widetilde{W}}{\mathrm{d}\tau} = \frac{\eta}{\tau(1-\tau)} + \left(\frac{\eta}{(1-\eta)\widetilde{\kappa}} - f''(\widetilde{\kappa})\widetilde{\kappa}\frac{1}{f(\widetilde{\kappa}) - f'(\widetilde{\kappa})\widetilde{\kappa}}\right)\frac{\mathrm{d}\widetilde{\kappa}}{\mathrm{d}\tau} \tag{B.9}$$

Note that $\frac{\mathrm{d}\ln\widetilde{W}}{\mathrm{d}\tau} > 0$ for $\eta > 1$ if $\widetilde{\kappa}^2 f''(\widetilde{\kappa})$ is small. This is a sufficient condition for workers to be absolutely better off. For the case of $\eta < 1$, this is not given, so we examine that case in Section 7: Figure 4 below shows that wages increase with τ in our calibrated model for moderate tax rates even when $\eta < 1$.

B.2 Proof of Proposition 2

Part 1 follows from monotonicity and the Intermediate Value Theorem.

For Part 2, the second-order differential inequality (Equation 22) on $\Phi(\tau)$ ensures that $\partial^2 \frac{\widetilde{S}}{\widetilde{K}}/\partial\tau^2 < 0$, i.e. wealth inequality is a strictly concave (and continuous) function of the tax rate. If $\lim_{\tau\to 1} \widetilde{S}/\widetilde{K} > 1$, the conclusion follows again by the Intermediate Value Theorem and the strict concavity implies there can be no more than one switch.

If instead $\lim_{\tau\to 1} \frac{\widetilde{S}}{\widetilde{K}} \leq 1$, note that wealth inequality has a unique maximum at $\Phi'(\tau^*)(\frac{\rho_c}{(1-\tau^*)} + \delta_k) = \Phi(\tau^*)(\frac{\rho_c}{(1-\tau^*)^2})$. Suppose this maximum occurs in the relevant range of $(0, 1)$. If the value of this maximum is smaller than 1, no switch to an Anti-Pasinetti regime occurs. If the value is greater or equal to 1, there is a switch for some tax rate and a switch back for a higher tax rate, applying the Intermediate Value Theorem again.[25] The reason is

[25]There is a singular case where the maximum value is just unity.

that by definition of strict concavity of a function, it can take a single value at most two times. If the maximum is not in the interval $(0, 1)$ there will be no switch to the Anti-Pasinetti regime.

B.3 Marginal change in the tax rate

Further, for a marginal change in the tax rate, it is also possible to characterise the effect of an increase in the tax rate on relative capital holdings in terms of the elasticity of substitution $\sigma(k)$ for general production functions, in a two class equilibrium:

Proposition 8. *$sgn(\mathrm{d}\ln(S/K)/\mathrm{d}\ln\tau) = sgn(1 - \Omega_K/\sigma(k))$. That is, the workers' share of wealth increases if the elasticity is greater than 1, holding factor shares constant.*

Proof.

$$\begin{aligned}\mathrm{sgn}(\mathrm{d}\ln(S/K)/\mathrm{d}\ln\tau) &= -\mathrm{sgn}(\mathrm{d}\ln(S/K)/\mathrm{d}\ln k)\\ &= \mathrm{sgn}(1 - \mathrm{d}\ln w/\mathrm{d}\ln k) = \mathrm{sgn}(1 - \Omega_K/\sigma(k))\end{aligned}$$

□

Hence, if the elasticity of substitution is greater than 1, wealth inequality is always decreased, *ceteris paribus*. If instead the elasticity is less than one, the effect of a marginal capital tax increase additionally depends on the factor share of capital.

C Convergence to steady state

For the version of the model used in Section 5, it can be shown that a (unique non-trivial) steady state exists whenever

$$\frac{\partial F}{\partial K}(K, J) = \frac{\rho_c}{(1-\tau)} + \delta_k. \tag{C.1}$$

It is well-known that the CES function does not fulfill this condition for all values of the elasticity. Here we use the parametrized version of our model from Section 7 to verify that a steady state exists for almost the entire range of the Pasinetti regime. Let again $J(P, L) = P_t^{\beta} L^{(1-\beta)}$. Figure 5 shows that for various values of β around its empirically plausible value of 0.2, the steady state to which the model converges exists for almost the entire Pasinetti range. Here we additionally simulated the parametrized version of Equation C.1. We checked that this finding holds for extensive variation of the further parameters (details available upon request).

Note that the steady state is saddle-point stable whenever it exists. This is because the dynamical system given by Equations (2), (3), (9) and (10) inherits the dynamics of the neoclassical growth model with public captial (Heijdra, 2009). To see this, note that Equation (9) only adds to the standard dynamics that in Equations (2) and (3) the interest rate is lower than if K^c was the only private capital input. This implies that there are no qualitative differences in the dynamics, only the steady-state value of K^c is smaller than the Keynes-Ramsey level of capital K (further details upon request).

D Derivations when public investment is labor-augmenting

D.1 Derivation of $\widetilde{S}/\widetilde{K}$

In this section we derive an explicit formula for the capital share of the workers $\widetilde{S}/\widetilde{K}$ (Equation 24).

We divide the expression for workers' saving (Equation 8) by total capital and then insert the firm's first-order conditions (Equations 13 and 12):

$$\frac{\widetilde{S}}{\widetilde{K}} = \frac{\widetilde{J}\widetilde{w}}{(2+\rho_w)\widetilde{K}} = \frac{(1-\alpha)\widetilde{J}^{\gamma}}{(2+\rho_w)\widetilde{K}\widetilde{Y}^{-(1-\gamma)}} = \frac{(1-\alpha)}{\alpha(2+\rho_w)}\Big(\frac{\rho_c}{1-\tau}+\delta_k\Big)\Big(\frac{\widetilde{J}}{\widetilde{K}}\Big)^{\gamma}. \tag{D.1}$$

Here we used that

$$w_t = \frac{\partial F(K_t, J_t)}{\partial J}. \tag{D.2}$$

To find an explicit solution for expression (D.1), solve the model for $\widetilde{K}/\widetilde{J}$. For this purpose, let $k = K/J$, and let $y = Y/J$. Then

$$y = \big(\alpha(k)^{\gamma} + (1-\alpha)1^{\gamma}\big)^{\frac{1}{\gamma}}. \tag{D.3}$$

From standard growth theory, we know that for any constant-returns-to-scale function

$r_t + \delta_k = \widetilde{Y}_K = \widetilde{y}'(k)$,

so that

$$\widetilde{y}'(k) = \frac{\rho_c}{1-\tau} + \delta_k. \tag{D.4}$$

To solve this, use that

$$\widetilde{y}'(\widetilde{k}) = \alpha\widetilde{k}^{\gamma-1}\big(\alpha\widetilde{k}^{\gamma} + (1-\alpha)\big)^{\frac{1-\gamma}{\gamma}}. \tag{D.5}$$

Substituting this into Equation (D.4) gives

$$\Big(\frac{1}{\alpha}(\frac{\rho_c}{1-\tau}+\delta_k)\Big)^{\frac{\gamma}{1-\gamma}} = \widetilde{k}^{-\gamma}(\alpha\widetilde{k}^{\gamma}+(1-\alpha)) = \alpha+(1-\alpha)\widetilde{k}^{-\gamma}. \qquad \text{(D.6)}$$

This is an equation that can be solved for $\widetilde{k}$,[26] as it is equivalent to

$$\frac{\widetilde{K}}{\widetilde{J}} = \widetilde{k} = \Big(\frac{1}{(1-\alpha)}\big((\frac{1}{\alpha}(\frac{\rho_c}{1-\tau}+\delta_k))^{\frac{\gamma}{1-\gamma}}-\alpha\big)\Big)^{\frac{-1}{\gamma}} \qquad \text{(D.7)}$$

This expression can be substituted into Equation (D.1) to obtain an explicit solution for the capital ratio:

$$\frac{\widetilde{S}}{\widetilde{K}} = \frac{(1-\alpha)}{\alpha(2+\rho_w)}\big(\frac{\rho_c}{1-\tau}+\delta_k\big)\Big(\frac{1}{(1-\alpha)}\big((\frac{1}{\alpha}(\frac{\rho_c}{1-\tau}+\delta_k))^{\frac{\gamma}{1-\gamma}}-\alpha\big)\Big). \qquad \text{(D.8)}$$

D.2 Properties of $\widetilde{S}/\widetilde{K}$

We determine the sign and zero of the derivative of $\widetilde{S}/\widetilde{K}(\tau)$. For this purpose, let $x(\tau) = (\rho_c/(1-\tau)+\delta_K)$, and note that $x'(\tau) = \rho_c(1-\tau)^{-2}$.

Then:

$$\frac{\widetilde{S}}{\widetilde{K}}(\tau) = \frac{1}{\alpha(2+\rho_w)}\big((\frac{1}{\alpha})^{\frac{\gamma}{1-\gamma}}(x(\tau))^{\frac{1}{1-\gamma}}-\alpha x(\tau)\big). \qquad \text{(D.9)}$$

Thus:

$$\begin{aligned}(\frac{\widetilde{S}}{\widetilde{K}})'(\tau) &= \frac{1}{\alpha(2+\rho_w)(1-\gamma)}(\frac{x(\tau)}{\alpha})^{\frac{\gamma}{1-\gamma}}x'(\tau)-\frac{x'(\tau)}{(2+\rho_w)} \\ &= (\frac{\rho_c}{(2+\rho_w)(1-\tau)^2})\Big(\frac{1}{\alpha(1-\gamma)}(\frac{1}{\alpha}(\frac{\rho_c}{(1-\tau)}+\delta_K))^{\frac{\gamma}{1-\gamma}}-1\Big) \qquad \text{(D.10)}\end{aligned}$$

We now compute the zero of the derivative by setting the second term of the product to 0:

$$\frac{1}{\alpha(1-\gamma)}(\frac{1}{\alpha}(\frac{\rho_c}{(1-\tau)}+\delta_K))^{\frac{\gamma}{1-\gamma}} = 1 \qquad \text{(D.11)}$$

This is equivalent to

$$(\frac{1}{\alpha}(\frac{\rho_c}{(1-\tau)}+\delta_K)) = (\alpha(1-\gamma))^{\frac{1-\gamma}{\gamma}} \qquad \text{(D.12)}$$

[26]Evidently solutions to Equation (D.7) could be complex if the term inside the exponent is negative. This reflects that outside of the Pasinetti regime, the model would not converge to a steady state given by the above equations as one class disappears. For further economic analysis, only the term's appearance in Equation (D.8) is used, which has an exponent equal to one. For $\sigma > 1$, our parametrized version of $F(J,K)$ thus fulfills the condition $\lim_{\tau\to 1} K/J = 0$ assumed in Proposition 1.

and further equivalent to

$$\frac{\rho_c}{(1-\tau)} = \alpha(\alpha(1-\gamma))^{\frac{1-\gamma}{\gamma}} - \delta_K. \tag{D.13}$$

Therefore,

$$\tau_z = 1 - \frac{\rho_c}{\alpha(\alpha(1-\gamma))^{\frac{1-\gamma}{\gamma}} - \delta_K}. \tag{D.14}$$

Further, replacing the equalities by inequalities, one can determine the sign of the derivative. This is, in general, dependent on the value of all relevant parameters. However, for non-restrictive parameter conditions, its sign can be determined for the economically relevant cases as follows.

Consider the above four equations as inequalities: First, note that for values $\gamma < 0$ the direction of the inequality changes from Equation (D.11) to (D.12). Second, noting that $\tau \in (0,1)$, there is also a change in the direction of the inequality from Equation (D.13) to (D.14) if

$$\alpha(\alpha(1-\gamma))^{\frac{1-\gamma}{\gamma}} > \delta_K. \tag{D.15}$$

For $|\gamma|$ small, it can be verified that this inequality holds for $\gamma < 0$, but not for $\gamma > 0$, for a wide parameter range for α and δ_k around their standard values of 0.38 and 0.7, respectively. For part of this parameter range, it also holds for large values of $|\gamma|$. Taken together, this means that the derivative is positive for $\tau < \tau_z$ and negative for $\tau > \tau_z$. Thus τ_z is a local maximum. The only economically relevant case that differs is for $\gamma < 0$ and $|\gamma|$ large ($\gamma < -0.95$ for the standard parametrization): in this case τ_z is a local minimum. However, for this case, $\tau_z > 1$ and thus $\widetilde{S}/\widetilde{K}$ is decreasing on $\tau \in (0,1)$.

Further, it can be verified, by inserting τ_z into the function, that the value of the maximum is given by

$$\frac{\widetilde{S}}{\widetilde{K}}(\tau_z) = -\frac{\alpha\gamma}{(2+\rho_w)}\big(\alpha(1-\gamma)\big)^{\frac{1-\gamma}{\gamma}}. \tag{D.16}$$

D.3 Proof of Proposition 5

To prove part (a), first note that

$$\lim_{\tau\to 1} \frac{\widetilde{S}}{\widetilde{K}}(\tau;\gamma) = -\infty.$$

This again follows from the algebra of limits, as

$$\lim_{\tau\to 1}\left(\left(\frac{1}{\alpha}(\frac{\rho_c}{1-\tau}+\delta_k)\right)^{\frac{\gamma}{1-\gamma}} - \alpha\right) = -\alpha.$$

It can be derived that $\widetilde{S}/\widetilde{K}(\tau)$ has a unique maximum in $\tau \in (0,1)$ for

$$\alpha(\alpha(1-\gamma))^{\frac{1-\gamma}{\gamma}} > \delta_K. \tag{D.17}$$

Else $\widetilde{S}/\widetilde{K}(\tau)$ is monotonically decreasing in (0,1) (see Appendix D.2).

First consider the case that a unique maximum exists. If the value of this maximum is below 1 (or outside of the range (0,1)), the Anti-Anti-Pasinetti case occurs, by the Intermediate Value Theorem, as $\widetilde{S}/\widetilde{K}(\tau)$ is continuous in $\tau \in (0,1)$. If instead the value of this maximum is above 1 and it is in the range $(0,1)$, the Anti-Pasinetti case occurs. However, if condition (D.17) is not fulfilled, the Anti-Anti-Pasinetti case occurs.

To prove part (b), note that the proof of monotonicity of $\widetilde{S}/\widetilde{K}(\tau;\gamma)$ in γ in the proof of Proposition 3 does not depend on γ being positive. $\gamma/(1-\gamma)$ is still a monotonically increasing function for $\gamma < 0$, given Assumption 1.

D.4 Proof of Corollary 6

It is established in Appendix D.2 that $\widetilde{S}/\widetilde{K}(\tau)$ has a unique maximum if $\alpha(\alpha(1-\gamma))^{\frac{1-\gamma}{\gamma}} > \delta_K$. (If this condition is not fulfilled, which is the case for $|\gamma|$ large, $\widetilde{S}/\widetilde{K}(\tau)$ has a minimum. The minimum, however, occurs for $\tau > 1$, and $\widetilde{S}/\widetilde{K}(\tau)$ can be shown to be decreasing within $\tau \in (0,1)$. See Appendix D.2 for details.) The value of this maximum is given by

$$\frac{\widetilde{S}}{\widetilde{K}}(\tau_z) = -\frac{\alpha\gamma}{(2+\rho_w)}\big(\alpha(1-\gamma)\big)^{\frac{1-\gamma}{\gamma}}. \tag{D.18}$$

Consider this value as a function of γ :

$$f(\gamma) = -\frac{\alpha\gamma}{(2+\rho_w)}\big(\alpha(1-\gamma)\big)^{\frac{1-\gamma}{\gamma}}. \tag{D.19}$$

The corollary is shown by proving the following properties, which are derived in Appendix D.2:

1. $f(\gamma)$ has a unique minimum with respect to γ at $\gamma = \ln(\alpha)$. It is monotonically increasing with respect to γ for $\gamma > \ln(\alpha)$.

2. $\lim_{\gamma\to 0+} f(\gamma) = +\infty$.

Assumption 1 implies that $f(\ln(\alpha)) < 1$ because one can deduce that there exists a γ with $\alpha(\alpha(1-\gamma))^{\frac{1-\gamma}{\gamma}} > \delta_K$. such that $\tau_z = 0$.[27]

[27]Condition $f(\ln(\alpha)) < 1$ is true if $-\frac{\alpha\ln(\alpha)}{(2+\rho_w)}\big(\alpha(1-\ln(\alpha))\big)^{\frac{1-\ln(\alpha)}{\ln(\alpha)}} < 1$, a condition that is satisfied by our standard parametrization (see Section 7) by a large margin.

The corollary is then deduced from the two properties in the following way: by the Intermediate Value Theorem, a value γ_1 exists, such that $f(\gamma_1) = 1$, since $f(\gamma)$ is continuous. This implies that for $\gamma < \gamma_1$, $f(\gamma_1) < 1$ and hence the Anti-Anti-Pasinetti case occurs. If $\gamma > \gamma_1$, then $f(\gamma_1) > 1$ and the economy is in an Anti-Pasinetti state.

Finally, note that Part (b) would not follow if it were the case that $\ln(\alpha) > \gamma_{\text{crit}}$ with γ_{crit} given by $\alpha(\alpha(1-\gamma_{\text{crit}}))^{\frac{1-\gamma_{\text{crit}}}{\gamma_{\text{crit}}}} = \delta_K$. In fact, it would violate the monotonicity of $\widetilde{S}/\widetilde{K}(\tau)$ throughout. Below we show why this cannot occur.

We now complete the proof of Part (a) of Corollary 5 by showing the two properties that

1. $f(\gamma)$ has a unique minimum with respect to γ at $\gamma = \ln(\alpha)$. It is monotonically increasing with respect to γ for $\gamma > \ln(\alpha)$.

2. $\lim_{\gamma \to 0+} f(\gamma) = +\infty$.

Regarding the first property, note that $f(\gamma)$ can be rewritten as

$$f(\gamma) = -\gamma\alpha^{1/\gamma}\Big(\frac{1}{(2+\rho_w)}(1-\gamma)^{\frac{1-\gamma}{\gamma}}\Big). \tag{D.20}$$

Let $g(\gamma) = -\gamma\alpha^{1/\gamma}$ and $h(\gamma) = 1/(2+\rho_w)\big((1-\gamma)\big)^{\frac{1-\gamma}{\gamma}}$. $h(\gamma)$ is monotonically increasing for all $\gamma > 0$, as is obtained from the fact that the function x^x is monotonically increasing. Further, it can be calculated that

$$\frac{\mathrm{d}g}{\mathrm{d}\gamma} = \alpha^{1/\gamma}\left(\frac{1}{\gamma}\ln(\alpha)\right). \tag{D.21}$$

This derivative equals zero for $\gamma = \ln(\alpha)$ and is positive for $\gamma > \ln(\alpha)$ and negative for $\gamma < \ln(\alpha)$. Since $f(\gamma)$ is the product of function g, which has a minimum at $\gamma = \ln(\alpha)$ and the monotonically increasing function h, it also has a minimum at $\gamma = \ln(\alpha)$. From this also follows that $f(\gamma)$ is monotonically increasing for $\gamma > \ln(\alpha)$.

Regarding the second property, factor $f(\gamma)$ into

$$f(\gamma) = \big(-\gamma\alpha^{1/\gamma}\big)\cdot\frac{1}{(1-\gamma)(2+\rho_w)}\cdot\big(1-\gamma\big)^{\frac{1}{\gamma}}. \tag{D.22}$$

Taking limits with respect to $\gamma \to 0$ from below, the second factor of this product tends to $1/(2+\rho_w)$. Note the third factor is equivalent to $\exp(1/x\ln(1-x))$. Applying L'Hôpital's rule to its exponent yields that this factor tends to e^{-1}.

It remains to consider the first term, $-\gamma\alpha^{1/\gamma}$. Substituting $\gamma = -1/y$ and applying L'Hôpital's rule to $(1/\alpha)^y/y$ as $y \to +\infty$ shows that this term tends to $+\infty$. This establishes the behavior at $\gamma = 0$ from below and completes the proof of Part (a).

We finally explain that given Assumption 1 it is always the case that $\ln(\alpha) < \gamma_{\text{crit}}$ as mentioned in the proof of Part (b) of Corollary 5. Recall that γ_{crit} is given by

$$\alpha(\alpha(1-\gamma_{\text{crit}}))^{\frac{1-\gamma_{\text{crit}}}{\gamma_{\text{crit}}}} = \delta_K. \tag{D.23}$$

Suppose for contradiction that $\ln(\alpha) > \gamma_{\text{crit}}$. Then by definition

$$\alpha(\alpha(1-\ln(\alpha)))^{\frac{1-\ln(\alpha)}{\ln(\alpha)}} > \delta_K. \tag{D.24}$$

Rearranging gives:

$$(1-\ln(\alpha))^{\frac{1}{\ln(\alpha)}} > \delta_K(1-\ln(\alpha)). \tag{D.25}$$

Noting that for $0 < \alpha < 1$, the right-hand side is bigger than δ_K and the left-hand side is smaller than α, one establishes a contradiction to Assumption 1 (b) in Section 5 of the main text.

E Variants of productivity-enhancing public investment

E.1 Public investment that enhances private capital

For the variant of the model in Section 6.1, assume now an explicit CES function between factors H and L in production, with the same parameters as in Equation (24), but H replacing K and L replacing J there. One finds that, entirely symmetrical to Equation (24) and by the method given in Appendix D.1, wealth inequality given by H_S/H ratio is given by:

$$\frac{\widetilde{H_S}}{\widetilde{H}} = \frac{(\widetilde{r}+\delta_K)(1-\alpha)}{\alpha(2+\rho_w)}(L/\widetilde{H})^{\gamma}. \tag{E.1}$$

This can be transformed to an expression with parameters only, as in Equation (24) and similarly to the method given in Appendix D.1:

$$\frac{\widetilde{H_S}}{\widetilde{H}} = \frac{(1-\alpha)}{\alpha(2+\rho_w)}\Big(\frac{\rho_c}{1-\tau}+\delta_k\Big)\Big(\frac{1}{(1-\alpha)}\big((\frac{1}{\alpha}(\frac{\rho_c}{1-\tau}+\delta_k))^{\frac{\gamma}{1-\gamma}}-\alpha\big)\Big). \tag{E.2}$$

So we have established:

Proposition 9. *For the case of public investment that augments a production factor as given by Equations (11-13) or Equations (12a-13a) and (25), it is irrelevant* for distributional outcomes *which factor is augmented.*

Proof. The right-hand side of Equation (E.2) is identical to that of Equation (24). □

Note the emphasis on distributional outcome in the proposition.

E.2 Capital-enhancing public investment with rents

Here we treat the case in which capital-enhancing public investment generates a rent, instead of augmenting the marginal product of private capital. Firms generate profits and we assume again that these are appropriated by the capitalists, so that households hold stocks of private capital, not augmented capital.

For a general production function with capital-enhancing public investment in the sense of Equation (25), one obtains Equation (26) again. Next, we analyze when the Anti-Pasinetti and Anti-Anti-Pasinetti may occur depending on substitution possibilities.

Therefore, in this subsection we analyze a production function of the nested CES type (instead of a single CES function and an unspecified subfunction, as with labor-enhancing public capital in Section 5). Assuming a general subfunction H will not help in this case because the interest rate is now given by $\partial F/\partial K$. Let production thus be given by:

$$Y_t = F_t(H_t, L) = (\theta H_t^{\mu} + (1-\theta)L^{\mu})^{(1/\mu)} \tag{E.3}$$

Public and private capital G and K are combined into generic capital H_t by means of a CES function:

$$H_t(K_t, P_t) = (\zeta K_t^{\eta} + (1-\zeta)P_t^{\eta})^{(1/\eta)}\,, \tag{E.4}$$

with $0 < \zeta < 1$ being the share parameter of private capital and $s = 1/(1-\eta)$ being the elasticity of substitution between private and public capital with $-\infty < \eta \leq 1$.

The ratio for wealth inequality in the steady state is then given by:

$$\frac{\widetilde{S}}{\widetilde{K}} = \frac{(\frac{\rho_c}{(1-\tau)} + \delta_K)(1-\theta)}{\theta\zeta(2+\rho_w)} \left(\frac{L}{H}\right)^{\mu} \left(\zeta + (1-\zeta)\left(\frac{\tau\rho_c}{(1-\tau)\delta_P}\right)^{\eta}\right). \tag{E.5}$$

An explicit expression for $(\frac{L}{H})^{\mu}$ can be determined by using the intensive form of the production function:

$$\left(\frac{L}{H}\right)^{\mu} = \frac{\left(\left(\frac{\frac{\rho_c}{(1-\tau)}+\delta_k}{\zeta\theta}\right)\left(\zeta + (1-\zeta)\left(\frac{\tau\rho_c}{\delta_p(1-\tau)}\right)^{\eta}\right)^{\left(\frac{\eta-1}{\eta}\right)}\right)^{\left(\frac{\mu}{1-\mu}\right)} - \theta}{(1-\theta)}. \tag{E.6}$$

To obtain Equation (E.5), insert Equations (12*b*) and (13*b*) into Equation (26) for the specified production structure, noting that

$$\left(\frac{H}{K}\right)^{\eta} = \left(\zeta + (1-\zeta)\frac{\widetilde{r}\tau}{\delta_P}\right)^{\mu}. \tag{E.7}$$

To further obtain the variable-free expression (E.6), proceed analogously to Appendix D.1 and let $h = H/L$, $y = Y/L$, etc. Computing the marginal product of capital in the intensive form, one finds that

$$\frac{\widetilde{r} + \delta_K}{k^{\eta-1}h^{1-\eta}} = \theta\zeta h^{\mu-1}\left(\theta h^{\mu} + (1-\theta)\right)^{\frac{1}{\mu}-1}. \tag{E.8}$$

Using Equation E.7 for the denominator on the left-hand side and solving for h, similar as in Appendix D.1, yields Equation (E.6).

In the derivations, it is additionally used that from Equation (10) in the steady state it follows that

$$\widetilde{P} = \frac{\tau\widetilde{r}\widetilde{K}}{\delta_P}. \tag{E.9}$$

We next examine whether the Anti-Pasinetti and Anti-Anti-Pasinetti regimes still can occur.

Proposition 10. *Let $\eta > 0$. For capital-enhancing public investment that does not augment factor prices, with the explicit production structure given by Equations (E.3-E.4) the Anti-Anti-Pasinetti case can occur for $\mu < 0$. For $\mu > 0$, only the Anti-Pasinetti regime can occur.*

The limiting behavior as $\tau \to 1$ of wealth inequality is now a more complicated combination in the space of the two elasticity parameters. We limit the treatment here to a partial result with high substitution possibilities between the two forms of capital. Proposition 10 is illustrated by Figure 6.

Proof. Let $\eta > 0$. Taking limits as $\tau \to 1$ in Equation (E.5) yields terms straightforwardly tending to positive infinity except those implicit in $\left(\frac{L}{H}\right)^{\mu}$ and given by Equation (E.6). It can be shown, by applying L'Hôpital's rule, that

$$\lim_{\tau\to 1}\left(\frac{L}{H}\right)^{\mu} = \begin{cases} +\infty & \text{if } \mu > 0, \\ -\theta/(1-\theta) & \text{if } \mu < 0 \end{cases} \tag{E.10}$$

and therefore

$$\lim_{\tau\to 1}\frac{\widetilde{S}}{\widetilde{K}} = \begin{cases} +\infty & \text{if } \mu > 0, \\ -\infty & \text{if } \mu < 0 \end{cases} \tag{E.11}$$

□

Setting $\mu = 0$ in Equation (E.5), one finds the steady-state wealth distribution for the special case in which the upper level of the CES-nest is Cobb-Douglas:

$$\frac{\widetilde{S}}{\widetilde{K}} = \frac{(\widetilde{r} + \delta_K)(1-\alpha)}{\alpha\zeta(2+\rho_w)} \left(\zeta + (1-\zeta)\left(\frac{\tau\widetilde{r}}{\delta_P}\right)^{\eta}\right). \tag{E.12}$$

From this expression, one can readily deduce the following special case:

Proposition 11. *With a nested CES production structure as assumed in Equations* (E.3) *and* (E.4) *and* $\mu = 0$

1. *the Anti-Anti-Pasinetti case cannot occur.*
2. *for every* $1 \geq \eta > 0$ *there exists a capital tax rate* τ_{lim} *from which on the economy switches from a Pasinetti to an Anti-Pasinetti state.*

In proving this proposition, we assume that Assumption 1 still holds, i.e. that $0 < \widetilde{S}/\widetilde{K}(\epsilon) < 1$ for ϵ small, which is the case for the meaningful parameter range and the second part is only meaningful if steady states exist (see discussion in Footnote 12).

Proof. Part 1 can be inferred directly from Equation (E.5): since $\widetilde{r}$, δ_K, δ_P, α, ζ, ρ_w are greater than zero, and $0 \leq \tau \leq 1$, the expression for $\widetilde{S}/\widetilde{K}$ is always strictly positive and has a strictly positive limit.

For Part 2, the idea of the proof is to show that $\widetilde{S}/\widetilde{K}(\tau)$ is monotonically increasing in τ, starting from a value lower than one and converging to infinity for $\tau \to 1$. The proof proceeds in two steps:

1. we show that $\lim_{\tau\to 1} \widetilde{S}/\widetilde{K}(\tau) = \infty$.
2. we show that $\widetilde{S}/\widetilde{K}(\tau)$ is monotonically increasing in $0 \leq \tau < 1$.

Regarding the first step, we insert the explicit expression for $\widetilde{r} = \rho_c/(1-\tau)$ and expand the products in Equation (E.5). This yields the following expression:

$$\begin{aligned}\frac{\widetilde{S}}{\widetilde{K}} &= \frac{(1-\alpha)}{\alpha\zeta(2+\rho_w)}\left[\left(\frac{\rho_c}{1-\tau} + \delta_K\right)\left(\zeta + (1-\zeta)\left(\frac{\tau\rho_c}{(1-\tau)\delta_P}\right)^{\eta}\right)\right] \\ &= \frac{(1-\alpha)}{\alpha\zeta(2+\rho_w)}\left[\lambda\zeta + (1-\zeta)\left(\left(\frac{\rho_c^{1+\eta}}{\delta_P^{\eta}}\mu\right) + \frac{\delta_K}{\delta_P^{\eta}}\nu\right)\right],\end{aligned}$$

with

$$\lambda(\tau) = \left(\frac{\rho_c}{1-\tau} + \delta_K\right),$$

$$\mu(\tau) = \frac{\tau^\eta}{(1-\tau)^{(1+\eta)}}$$

and

$$\nu(\tau) = \left(\frac{\tau\rho_c}{(1-\tau)}\right)^\eta.$$

It can be inferred from these equations directly that for $\tau \in (0,1)$

$$\lim_{\tau\to 1} \lambda(\tau) = \lim_{\tau\to 1} \mu(\tau) = \lim_{\tau\to 1} \nu(\tau) = \infty,$$

which implies that $\lim_{\tau\to 1} \widetilde{S}/\widetilde{K}(\tau) = \infty$.

Regarding the second step, it remains to show that $\widetilde{S}/\widetilde{K}(\tau)$ is monotonically increasing for all $\tau \in (0,1)$.

Since we only consider $\eta > 0$, that is, the case of elasticities between public and private capital greater than or equal to one, this is straightforward to show: $\widetilde{S}/\widetilde{K}(\tau)$ is the sum of the monotonically increasing functions $1/(1-\tau)$, $\tau^\eta/(1-\tau)^{(1+\eta)}$ and $(\tau/(1-\tau))^\eta$, multiplied by positive constants. All these functions are monotonically increasing for $\eta > 0$. This implies that the function $\widetilde{S}/\widetilde{K}(\tau)$ is monotonically increasing.

Since we assume that $0 < \widetilde{S}/\widetilde{K}(0,\gamma) < 1$ and we showed that $\widetilde{S}/\widetilde{K}(\tau)$ is monotonically increasing in τ and $\lim_{\tau\to 1} \widetilde{S}/\widetilde{K}(\tau) = \infty$, it follows from the Intermediate Value Theorem that for a given $0 < \eta < 1$, there exists a $\tau_{lim} \in (0,1)$, with $\widetilde{S}/\widetilde{K}(\tau_{lim}) = 1$. For this τ_{lim} the Pasinetti regime changes into an Anti-Pasinetti regime.

□

Note that the case $\eta < 0$, that is substitution elasticity $s < 1$, is not treated in Propositions 10 and 11. The reason is that one can show that for small tax rates and $\mu = 0$, capitalists vanish because the limit of S/K tends to infinity as the tax rate approaches 0. This is not a surprising finding: The assumption that private and public capital are highly complementary implies that, for low taxes, the value of private capital is strongly diminished and capitalists' income is decreased. However, as this setting only considers good substitutability between capital and *labor,* this increases wages and explains how for low tax rates the Anti-Pasinetti case can reappear.

E.3 The case of perfect substitutability between private and public capital

Finally consider the special case of Proposition 11 of a perfect elasticity of substitution between public and private capital, for which the value of the Anti-Pasinetti tax rate can be calculated explicitly. Set $\zeta = 0.5$ and $\eta = 1$ in Equation (E.12) and assume $\delta_K = \delta_P = \delta$.

Proposition 12. *For the case of a perfect elasticity of substitution between public and private capital, there exists a capital tax rate τ_{lim} at which the Pasinetti regime changes to the Anti-Pasinetti regime. For equal depreciation across capital stocks, this tax rate is given by the $\tau_{lim}^{1,2}$ which is in the economically meaningful range of* $(0, 1)$*:*

$$\tau_{lim}^{1,2} = \frac{\frac{\rho_c^2}{\delta} - 2(\delta - \frac{1}{x}) \pm \frac{\rho_c}{\delta}\sqrt{1 + 4\frac{\delta}{x}}}{2\frac{1}{x} - \delta + \rho_c}. \tag{E.13}$$

Proof. For the case at hand, wealth inequality is given by:

$$\frac{\widetilde{S}}{\widetilde{K}} = \frac{(\widetilde{r} + \delta_K)(1-\alpha)}{\alpha(2+\rho_w)}\left(1 + \left(\frac{\tau\widetilde{r}}{\delta_P}\right)\right). \tag{E.14}$$

To determine the capital tax rate τ_{lim} at which the regime changes from a Pasinetti to an Anti-Pasinetti state we set $S/K = 1$ in Equation (E.14). Let $x = \frac{(1-\alpha)}{\alpha(2+\rho_w)}$, then:

$$1 = x\left(\frac{\rho_c}{(1-\tau_{\text{lim}})} + \delta_K\right)\left(1 + \left(\frac{\tau_{\text{lim}}}{\delta_P}\frac{\rho_c}{(1-\tau_{\text{lim}})}\right)\right). \tag{E.15}$$

Solving for τ yields the following quadratic equation:

$$(\tau_{\text{lim}})^2 \underbrace{\left[\frac{1}{x} - \delta_k + \rho_c\frac{\delta_K}{\delta_P}\right]}_{a} + \tau_{\text{lim}} \underbrace{\left[\rho_c\left(1 - \frac{\delta_K}{\delta_P} - \frac{\rho_c}{\delta_P}\right) + 2\left(\delta_k - \frac{1}{x}\right)\right]}_{b} + \underbrace{\left[\frac{1}{x} - \delta_k - \rho_c\right]}_{c} = 0. \tag{E.16}$$

Therefore,

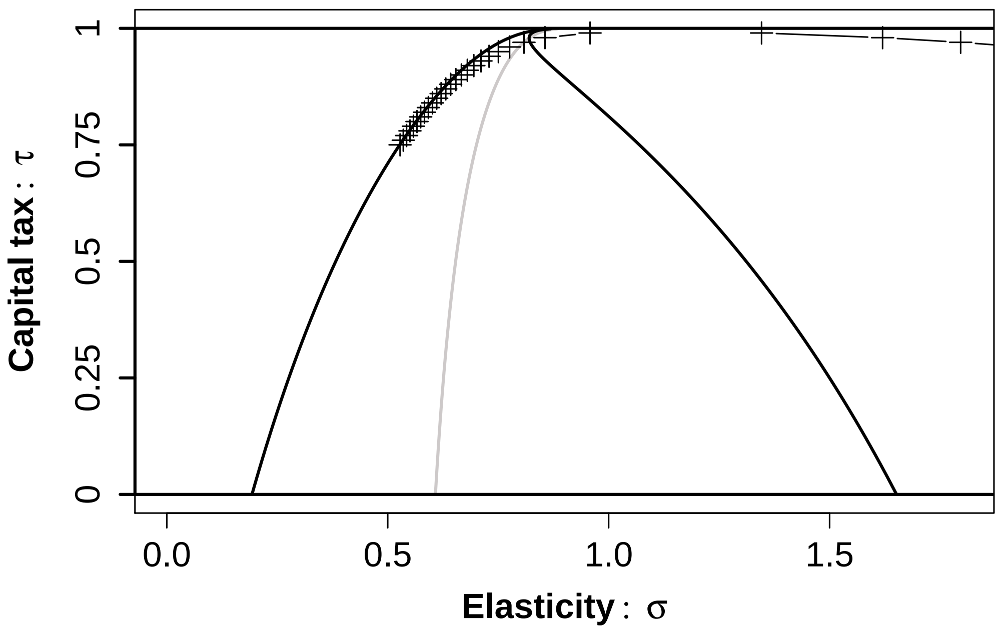
$$\tau_{\text{lim}}^{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{E.17}$$

Set $\delta = \delta_k = \delta_P$, to obtain the expression in Proposition 12.

□

Equation (E.13) permits to study the dependency of the critical tax rate on parameters. For example, one finds that it increases monotonically in the pure time preference rate of the capitalists, while it decreases monotonically in the workers' time preference rate (details available upon request).[28]

[28]For the standard parametrization (see Section 7) the economically meaningful tax rate at which the economy switches from a Pasinetti to an Anti-Pasinetti state is 54 %. The sensitivity to changes in the capitalists' time preference rate is much stronger than the sensitivity to changes in the workers' time preference rate.



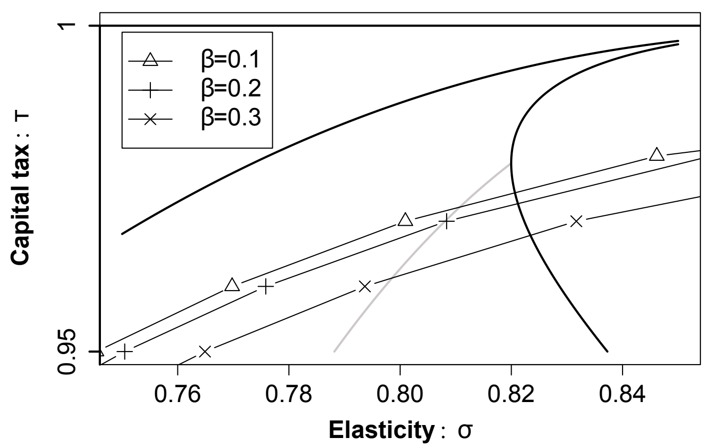


Figure 5: Equilibrium outcomes of the model with labor-enhancing public investment as a function of the elasticity of substitution between capital and labor σ and the capital tax rate τ as in Section 7, Figure 3. In the upper panel, above the line consisting of crosses no steady state exists for $\beta = 0.2$. The lower panel visualizes differences in the existence of steady states for different values of β : No steady state exists above these lines.

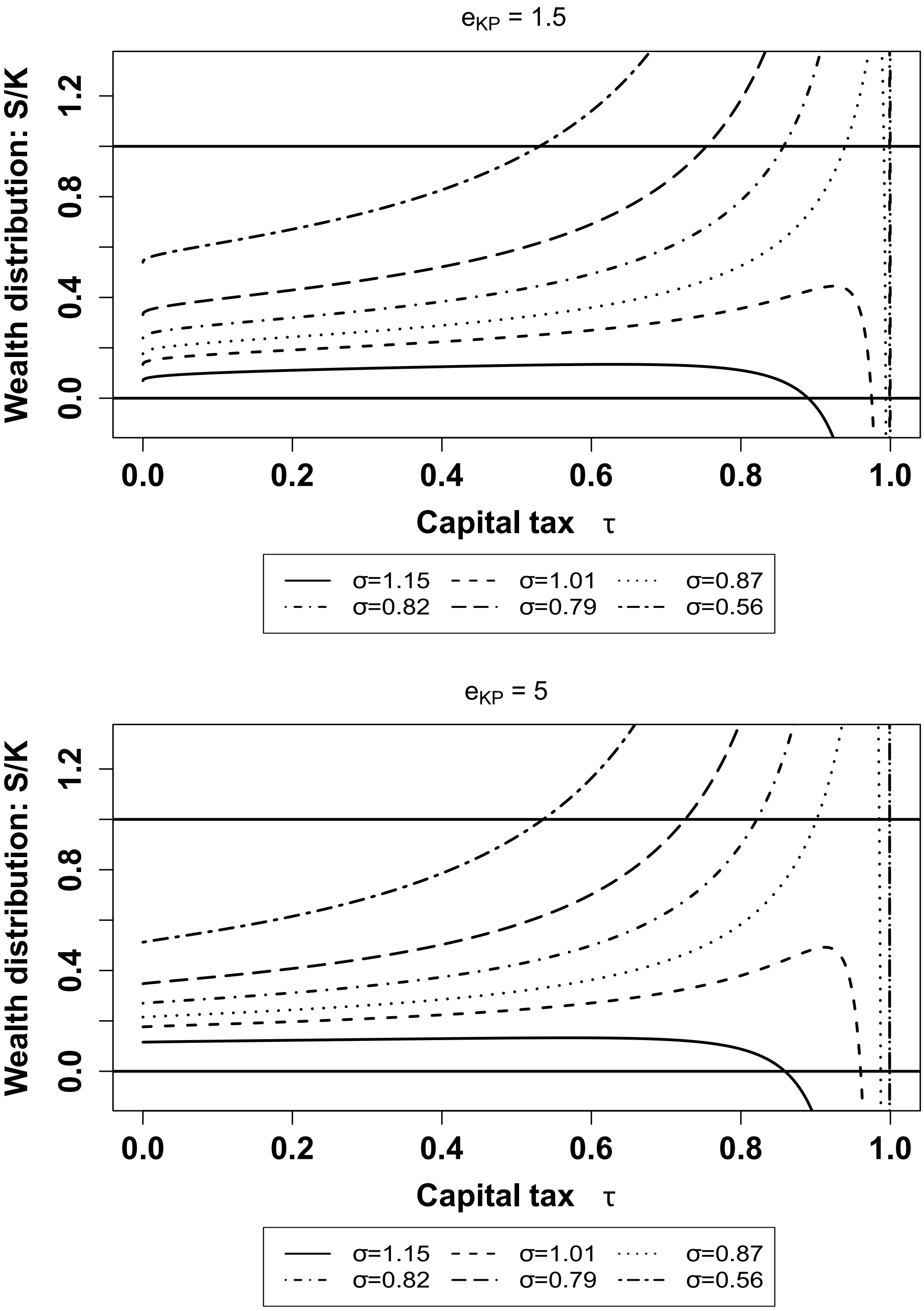


Figure 6: Wealth inequality as a function of the capital tax rate for various elasticities of substitution between aggregate capital and labor as an illustration of Proposition 8. The upper panel illustrates these for a value of the elasticity of substitution between private and public capital of 1.5. This value is 5 in the lower panel. Values above 1 and below 0 are not economically meaningful, but used in demonstrations.