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Zahra Dayani; Fatemeh Parastesh ; Fahimeh Nazarimehr ; Karthikeyan Rajagopal ; Sajad Jafari ;
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Zahra Dayani,[1] Fatemeh Parastesh,[1] Fahimeh Nazarimehr,[1] Karthikeyan Rajagopal,[2] Sajad Jafari,[1,3] Eckehard Schöll,[4,5,6,a)] and Jürgen Kurths[6,7]

AFFILIATIONS

[1] Department of Biomedical Engineering, Amirkabir University of Technology (Tehran Polytechnic), Tehran 159163-4311, Iran
[2] Centre for Nonlinear Systems, Chennai Institute of Technology, Chennai 600069, India
[3] Health Technology Research Institute, Amirkabir University of Technology (Tehran Polytechnic), Tehran 159163-4311, Iran
[4] Institut für Theoretische Physik, Technische Universität Berlin, Hardenbergstrasse 36, 10623 Berlin, Germany
[5] Bernstein Center for Computational Neuroscience Berlin, Humboldt-Universität, 10115 Berlin, Germany
[6] Potsdam Institute for Climate Impact Research, Telegrafenberg A 31, 14473 Potsdam, Germany
[7] Department of Physics, Humboldt University Berlin, Berlin 12489, Germany

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[a)] **Author to whom correspondence should be addressed:** schoell@physik.tu-berlin.de

ABSTRACT

In this paper, we propose a time-varying coupling function that results in enhanced synchronization in complex networks of oscillators. The stability of synchronization can be analyzed by applying the master stability approach, which considers the largest Lyapunov exponent of the linearized variational equations as a function of the network eigenvalues as the master stability function. Here, it is assumed that the oscillators have diffusive single-variable coupling. All possible single-variable couplings are studied for each time interval, and the one with the smallest local Lyapunov exponent is selected. The obtained coupling function leads to a decrease in the critical coupling parameter, resulting in enhanced synchronization. Moreover, synchronization is achieved faster, and its robustness is increased. For illustration, the optimum coupling function is found for three networks of chaotic Rössler, Chen, and Chua systems, revealing enhanced synchronization.

In many diverse applications of complex networks, enhancing synchronization is a crucial problem. Previous research has attempted to improve synchronization using various techniques, including time-varying coupling functions or coupling strength. Based on the master stability approach, the current research proposes a time-varying coupling function for achieving enhanced synchronization. Compared to constant single-variable couplings, the optimal coupling function, which only includes one variable in the coupling at a time, enables synchronization at lower coupling strengths. As a result, synchronization cost is reduced. The optimal coupling function is revealed for three classical chaotic systems, Rössler, Chen, and Chua. Additionally, the optimal time-varying coupling function is used in numerical simulations of ring networks, and the findings are discussed.

I. INTRODUCTION

Networks of dynamical systems are of great importance in different fields, ranging from physics and biology to engineering and social sciences.[1,2] The collective behavior of the networks depends on the configuration of the coupling between the dynamical systems of the nodes. A prominent collective behavior in networks is synchronization.[3–7] The synchronized behavior plays essential roles in different areas,[8–10] such as in normal neuronal processes in cognitive tasks,[11] pathological brain functions,[12,13] food web dynamics in ecological systems,[14] and power grids.[15,16] Consequently, many studies have been devoted to synchronization and investigating its stability. A common and efficient method for studying the stability of synchronization in diffusively coupled identical oscillators is the master stability function (MSF).[17] Using this method, the conditions that guarantee stability of synchronization can be achieved.

Some modifications of the MSF approach can also be applied to special coupling schemes, such as delayed coupling and time-varying coupling.[18–23]

Most studies on synchronization have focused on enhancing and optimizing synchronization.[24–27] For example, it has been revealed that an enhanced synchronization is achieved when the network is directed and weighted and depends on the mean degree.[28–30] Banerjee *et al.*[31] reported that inducing heterogeneity by adding a mismatched oscillator in the network of identical oscillators can help in reaching synchronization for smaller couplings. Fan *et al.*[25] found that a proper selection of phase lag modulation can lead to optimum synchronization. Sevilla-Escoboza[27] tried a multi-variable coupling to find the optimum coupling for maximizing the stability of synchronization. Taher *et al.*[32] could achieve an enhanced synchronization by applying the time-delayed feedback control to a subset of nodes involved in disturbing synchronization. Martineau *et al.*[26] showed that the existence of anticorrelated noise patterns can enhance synchronization. Hazrati *et al.*[33] compared the synchronizability of different network topologies and their structural properties by varying the number of links. They represented that for a low number of links, the best synchronization is related to the network with the lower average and variance of path length, and for the larger number of links, it is associated with the lower clustering coefficient.

A few synchronization analyses have been done by considering time-varying topologies and couplings.[34–37] Belykh *et al.*[38] studied synchronization in small-world networks with blinking random short-cuts. Stilwell *et al.*[39] represented that the stability of synchronization in networks with blinking links is equivalent to the average network when blinking is fast. The average network is a static network whose connections are the time average of blinking connections. Jeter and Belykh[40] found some intermediate on/off frequency ranges with stable synchronization, in which the average network was unstable. Kohar *et al.*[35] considered networks with stochastically rewiring the links and showed that in time-varying networks, the stability range is increased and the synchronization is achieved faster. Hagos *et al.*[37] reported synchronization transitions in phase oscillators by considering time-varying coupling functions. Parastesh *et al.*[41] studied a periodic blinking coupling between variables in networks with constant topology. They showed that this coupling can decrease the synchronization cost compared to constant single-variable coupling.

In previous studies, most of the time-dependent generalizations have been devoted to enhancing synchronization through adaptive coupling strengths or time-varying connections, and less attention has been paid to the coupling function. This paper aims to find a time-varying coupling function that enhances synchronization in networks of diffusively coupled systems. The coupling is considered to be a single-variable but varies between different variables in time (the coupling variables change over time adaptively). The procedure for finding the appropriate time-varying coupling function that relies on the master stability approach is described in Sec. II. Then, in Sec. III, the method is applied to three paradigmatic networks of the Rössler, Chen, and Chua systems. It is shown that by using the proposed time-varying coupling function, synchronization is enhanced compared to time-constant single-variable coupling functions. Finally, our conclusions are given in Sec. IV.

II. THE NETWORK AND TIME-VARYING COUPLING FUNCTION

A network of N identical oscillators with a linear diffusive time-varying coupling function can be described by

$$\dot{X}_i(t) = F(X_i(t)) - \sigma \sum_{j=1}^{N} G_{ij} H_t(X_j(t)), \tag{1}$$

where X_i is the m-dimensional state vector of the ith oscillator, $i = 1, \dots, N; F: \mathbb{R}^m \to \mathbb{R}^m$ describes the dynamics of the uncoupled system; σ denotes the overall coupling strength; and $G_{N\times N}$ is the Laplacian matrix of the connection topology. The coupling function is denoted by $H_t(X_j(t))$. Since the coupling is considered linear, it can be rewritten as $H_t(X_j(t)) = h(t)X_j(t)$, where $h(t)$ is a time-varying m–m matrix. $h_{ij} = 1$ describes $j \to i$ coupling; i.e., the coupling is in the jth state variable and added to the ith state variable. Here, we assume that at each time interval, only one matrix element of h is unity and the others are zero. The aim is to find a time-varying coupling function for which the synchronization stability region is expanded; hence, it enhances synchronization.

A. Stability of the synchronization

The stability of the synchronization manifold ($X_1 = X_2 = \cdots = X_s$) for a network can be found by the master stability approach. In this method, a small perturbation is applied to the synchronous manifold as $\delta X_i = X_s - X_i$. If all perturbations decay to zero, then the synchronous manifold is stable. The dynamical equations of perturbations, which are known as the variational equations, can be found by substituting δX_i in Eq. (1) and linearizing around the synchronous solution as follows:

$$\delta\dot{X}_i(t) = DF(X_s)) - \sigma \sum_{j=1}^{N} G_{ij} DH(X_s)\delta X_j, \tag{2}$$

where $DF(X_s)$ and $DH(X_s)$ denote the Jacobian of F and H at the synchronous manifold. Since the coupling is linear, we have $DH(X_s) = h(t)$. Hence, Eq. (2) can be rewritten as

$$\begin{aligned} \delta\dot{X}_i(t) &= DF(X_s)) - \sigma \sum_{j=1}^{N} G_{ij} h(t)\delta X_j \\ &= DF(X_s) - \sigma h(t) \sum_{j=1}^{N} G_{ij}\delta X_j. \end{aligned} \tag{3}$$

By diagonalization of the Laplacian matrix G, the decoupled form of the variational equations can be found,

$$\dot{\eta}_i(t) = [DF(X_s)) - \sigma h(t)\lambda_i]\eta_i, \tag{4}$$

where $\boldsymbol{\eta}(t) = (\eta_1, \eta_2, \dots, \eta_N)$ is an Nm-dimensional vector since each variable η_i is m-dimensional. $\boldsymbol{\eta}$ is obtained from the transformation of the perturbations under $\boldsymbol{\eta} = Q^{-1}\delta\boldsymbol{X}$ in which the matrix Q is constructed from the eigenvectors of G. The eigenvalues of G are denoted by λ_i, $i = 1, \dots, N$, where for $i = 1, \lambda_1 = 0$, and the solution is along the synchrony manifold and for $i = 2, \dots, N$, it is transverse to the synchrony manifold. The transverse stability of the synchronization manifold follows from Eq. (4) for $i = 2, \ldots, N$ by calculating the maximum Lyapunov exponent (Λ) as a function of the parameters λ_i, known as the master stability function (MSF) $\Lambda(\lambda)$. If $\Lambda < 0$ holds for a given $K = \sigma\lambda$, then the solutions of the variational equations asymptotically decay and consequently, the synchronization manifold is locally stable.

B. Optimal time-varying coupling function

For a given value of $K = \sigma\lambda$, the MSF depends on the coupling function (h). When it is chosen as constant in time ($h(t) = h$), the maximum Lyapunov exponent of Eq. (4) is usually computed for a long time interval. For an m-dimensional dynamical system, one can define $m \times m$ single-variable coupling schemes (each with only one non-vanishing matrix element of h, normalized to unity without loss of generality). Among these coupling schemes, one, which we refer to as the best constant coupling, has the largest stability region and results in synchronization at the lowest coupling strength. Our hypothesis is that the synchronization can be enhanced by considering short time intervals and finding the proper coupling scheme among $m \times m$ couplings for each time interval. As the MSF method calculates the maximum Lyapunov exponent of variational equations [Eq. (4)], the proper coupling can be found by comparing the local Lyapunov exponent of variational equations for all coupling schemes. Then, the proper coupling scheme is the one with the lowest local largest Lyapunov exponent. Consequently, the coupling function obtained by the proper coupling schemes in each time interval is optimal. The optimal coupling function leads to enhanced synchrony even better than the best constant coupling.

For a trajectory, the local Lyapunov exponents are obtained by dividing the total trajectory into small segments and computing the Lyapunov exponents of the segments. It is noted that here, we always analyze the largest Lyapunov exponent. The time interval considered for finding the optimal coupling function refers to the time of calculating the local Lyapunov exponent. According to standard procedures,[42] this time step should not be too large, but also not too small. If the time interval considered is large, although the local Lyapunov exponents are calculated correctly, the change in the coupling occurs slowly and cannot improve synchronization. Hence, the time interval should be chosen small enough to achieve fast blinking relative to the time scale of the oscillatory system dynamics. On the other hand, a very small time interval increases the computational cost. Hence, there should be a compromise between the fast blinking and the computational cost. Under this aspect, we have done many simulations and found that $T = 0.5$ is the best time step for calculating the Lyapunov exponents. We have also checked in the two-parameter plane (T, K) that the largest Lyapunov exponent does not depend on T in some range around $T = 0.5$ at fixed K.

According to the above descriptions, the steps for finding the optimum time-varying coupling function that results in enhanced synchronization are as follows:

1. An appropriate time interval is considered for varying the coupling function.
2. For each time interval, all possible constant coupling functions are considered. ($m \times m$ coupling functions exist, each with only one non-zero element.)
3. The local Lyapunov exponent of Eq. (4) is computed for all constant coupling functions.
4. The coupling function, which has the lower local Lyapunov exponent, is selected for this time interval.

Consequently, the best coupling function for each time interval can be found. The resulting coupling function ($h(t)$) leads to an MSF for the chosen value of $K = \sigma\lambda$ with lower values than each of the $m \times m$ constant coupling functions.

C. Applicability of the method

In this subsection, we discuss the features of the theory and describe how it can be applied to real systems.

The variational equation [Eq. (4)] depends on the synchronous manifold (X_s), which is the same as the trajectory of a single uncoupled system for linear diffusive coupling. In the proposed method, the local Lyapunov exponent of the variational equations should be calculated, which is dependent on the trajectory and its initial condition. Accordingly, the proper coupling scheme in each time interval relies on the trajectory. Therefore, using different initial conditions for the trajectory, different time-varying optimal coupling functions are obtained. However, the general form and the number of times each coupling is turned on are almost the same in all coupling functions. (It has been observed in 100 simulations with different initial conditions.)

According to the above description, the optimal time-varying coupling function is found from one trajectory with a well-defined initial condition. However, in a network consisting of N oscillators, there are N initial conditions, each resulting in a different time-varying coupling function for the respective oscillator. Therefore, to apply the theory to coupled systems, they must first reach a synchronized state to be in an equivalent state (which can be formally considered a similar initial condition). Then, the optimal time-varying coupling function can be obtained and applied to the network. By using the obtained time-varying coupling function, the systems can remain synchronous under a lower coupling strength than the best for constant coupling.

III. NUMERICAL ILLUSTRATION

This section presents some basic examples to show that the defined time-varying coupling function can enhance synchronization. To this aim, three paradigmatic chaotic systems are selected: Rössler,[43] Chen,[44] and Chua.[45] The dynamical equations of these systems are as follows:

Rössler system:

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + \alpha y, \\ \dot{z} &= \beta + (x - \gamma)z, \end{aligned} \tag{5}$$

where $\alpha = \beta = 0.2$ and $\gamma = 9$.

Chen system:

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= (c - a - z)x + cy, \\ \dot{z} &= xy - \beta z, \end{aligned} \tag{6}$$

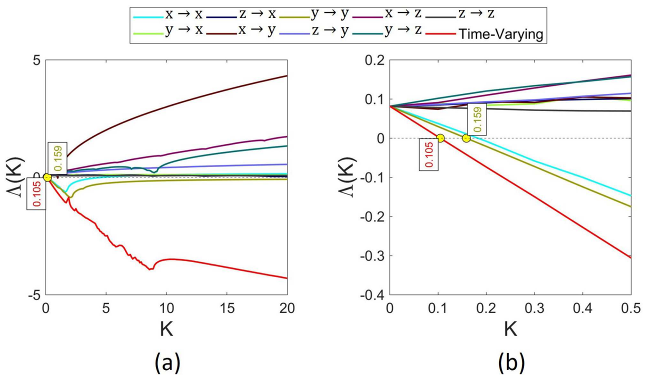

FIG. 1. The largest Lyapunov exponent (Λ) of the variational equations for the Rössler system vs the coupling parameter K. (a) A large range of K and (b) a blow-up of part (a) in a smaller range of K. The red color refers to the optimal time-varying coupling, and other colors show constant couplings, where $j \rightarrow i$ coupling means that the coupling is in the j state variable and added to the i state variable.

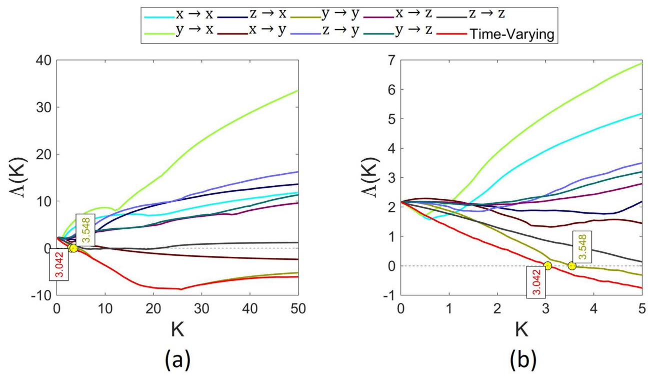

FIG. 2. The largest Lyapunov exponent (Λ) of the variational equations for the Chen system vs coupling parameter K. (a) A large range of K and (b) a blow-up of part (a) in a smaller range of K. The red color refers to the optimal time-varying coupling, and other colors show constant couplings, where $j \rightarrow i$ coupling means that the coupling is in the j state variable and added to the i state variable.

where $a = 35, c = 28$, and $\beta = \frac{8}{3}$.

Chua system:

$$\begin{aligned} \dot{x} &= \alpha[y - x + f(x)], \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y - \gamma z, \\ f(x) &= \begin{cases} -bx - a + b, & x > 1, \\ -ax, & |x| < 1, \\ -bx + a - b, & x < -1, \end{cases} \end{aligned} \tag{7}$$

where $\alpha = 10, \beta = 14.87, \gamma = 0, a = -1.27$, and $b = -0.68$.

For all of the systems, the variational equations are derived. Then, as described in Sec. II, time intervals are considered (here, $T = 0.5$, which is small compared to the timescale of the intrinsic oscillatory dynamics). The local Lyapunov exponents of the variational equations are calculated for constant coupling functions. The coupling function with the lowest local Lyapunov exponent is chosen for the corresponding time interval. Finally, a time-varying coupling function is obtained. For this case, the largest Lyapunov exponent of the variational equations, based upon the obtained time-varying coupling function, is lower than that for constant coupling functions.

Figures 1–3 represent the master stability function, i.e., the largest Lyapunov exponent (Λ) of the variational equations with time-varying as well as constant coupling functions vs the coupling parameter $K = \sigma\lambda$ for the Rössler, Chen, and Chua systems, respectively. The left parts show $\Lambda(K)$ in a larger range of K, and the right parts are zoomed-in views of part (a) in a smaller range of K to display the difference between the zero-crossing points better. The largest Lyapunov exponent referring to the time-varying coupling function is shown in red color. We observe that the MSF curve with a time-varying coupling function is clearly lower than the constant coupling functions. Moreover, it has a smaller zero-crossing point, where the synchronous solution becomes stable. For the Rössler system, the best synchronization with constant coupling occurs for $K > 0.159$ when coupled with $y \rightarrow y$ coupling. In contrast, time-varying coupling results in stable synchronization for $K > 0.105$. Similarly, for the Chen system (Fig. 2), the best synchronization relates to $y \rightarrow y$ coupling with the zero-crossing stability threshold at $K = 3.548$. However, the time-varying coupling decreases this threshold to $K = 3.042$. Finally, for the Chua system (Fig. 3), $x \rightarrow y$ coupling leads to stable synchronization for $K > 0.44$, while the time-varying coupling function enables synchronization for $K > 0.37$. In conclusion, the obtained time-varying coupling function results in strongly enhanced stability of synchronization.

To provide more detail on the coupling function in each time interval of length $T = 0.5$, the trajectories of three chaotic systems

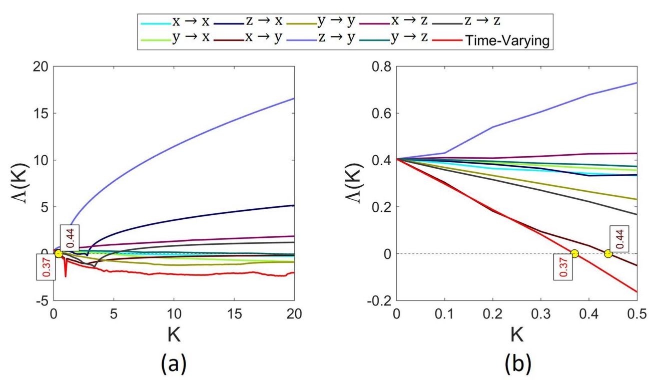

FIG. 3. The largest Lyapunov exponent (Λ) of the variational equations for the Chua system vs coupling parameter K. (a) Large range of K. (b) A blow-up of part (a) in a smaller range of K. The red color refers to the optimal time-varying coupling, and other colors show constant couplings, where $j \rightarrow i$ coupling means that the coupling is in the j state variable and added to the i state variable.

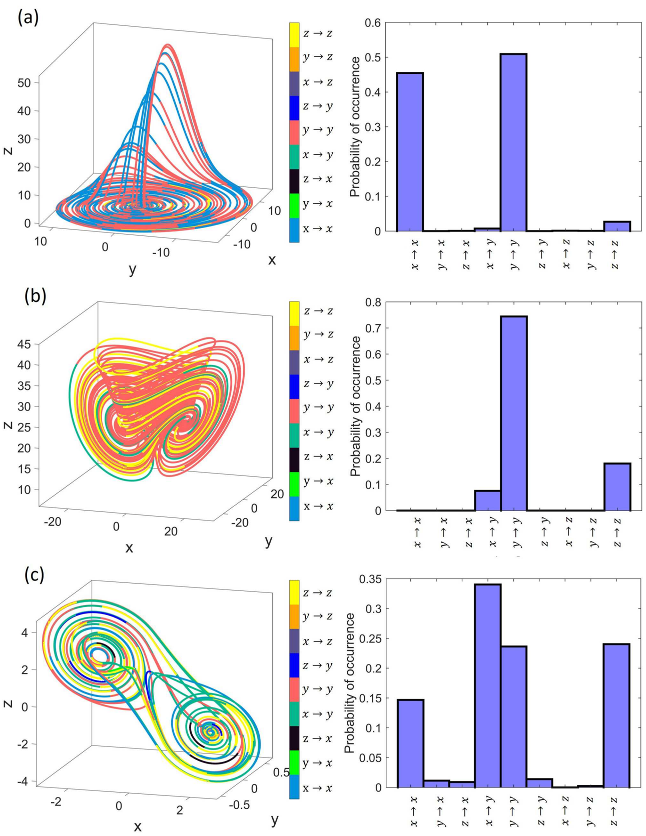

FIG. 4. The left column shows the optimal coupling function in each time interval by color on the system trajectory. The right column shows the probability of occurrence of each coupling. (a) Rössler system in Eq. (5) with $\alpha = \beta = 0.2$ and $\gamma = 9$ ($\sigma = 5$). (b) Chen system in Eq. (6) with $a = 35, c = 28$, and $\beta = 8/3$ ($\sigma = 50$). (c) Chua system in Eq. (7) $\alpha = 10, \beta = 14.87, a = -1.27$, and $b = -0.68$ ($\sigma = 5$). The colors of the trajectory represent the optimal coupling for the corresponding time interval. The couplings $x \to x$, $y \to x$, $z \to x$, $x \to y$, $y \to y$, $z \to y$, $x \to z$, $y \to z$, and $z \to z$ are shown by light blue, light green, black, dark green, red, dark blue, purple, orange, and yellow, respectively, where $j \to i$ coupling means that the coupling is in the j state variable and added to the i state variable. The Rössler, Chen, and Chua trajectories are plotted for 270, 450, and 900 time units, respectively. The initial points of the trajectory in parts (a)–(c) are $(0, 0, -1)$, $(7.72, 9.29, 21.19)$, and $(-1.89, -0.42, 2.17)$.

are shown in Fig. 4 (left panel) and the optimal coupling in the time intervals is shown by different colors. One can clearly see that during one oscillation cycle, several time intervals of different coupling occur. It can also be observed that one coupling scheme is usually optimal in some successive time intervals. The choice of coupling for each time interval depends on the value of the local Lyapunov exponent, which is relevant to the eigenvalues of the linearized variational system $(DF - Kh)$. In fact, the Jacobian of the system varies according to the points of the trajectory. Also, K can be involved in this system in nine different forms according to the nine coupling schemes. Hence, the eigenvalues of this system differ for different time intervals and different coupling schemes, resulting in different

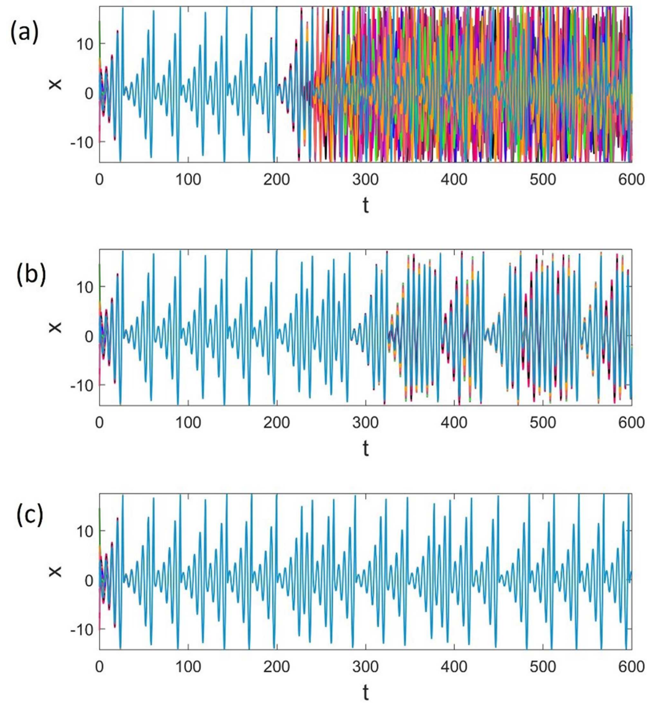

FIG. 5. The time series of 20 Rössler systems [Eq. (5)] coupled in a locally coupled ring network. At first, the three-variable coupling with $\sigma = 1.74$ as strong coupling is applied until $t = 100$. (a) There is no coupling between the oscillators after $t = 100$. (b) The systems are coupled via the best constant coupling ($y \to y$) after $t = 100$ with the coupling strength $\sigma = 1.23$. (c) The systems are coupled via the optimal time-varying coupling after $t = 100$ with the coupling strength $\sigma = 1.23$. The parameters of the Rössler systems are $\alpha = \beta = 0.2$ and $\gamma = 9$. The time series are obtained using 4th-order Runge–Kutta method with a time step of 0.01.

local Lyapunov exponents. It can be seen, e.g., in Fig. 4(a) for the Rössler system that the optimum coupling generally involves those variables, which change most rapidly along each particular section of the trajectory. For instance, during the rise of the z-variable from the (x, y) spiral plane with increasing y, the $y \to y$ coupling dominates, while subsequently during re-injection with decreasing x, mostly the $x \to x$ coupling dominates.

Figure 4 also represents that for each system, some of the couplings are turned on more frequently than others. Figure 4(a) shows that for the Rössler system, the optimal coupling mostly consists of $x \to x$ and $y \to y$ couplings, while, for the Chen system [Fig. 4(b)], the couplings $y \to y$ and $z \to z$ are turned on more frequently. For the Chua system [Fig. 4(c)], the switching frequency of couplings $x \to x$, $x \to y$, and $z \to z$ is higher. The probability of occurrence of each coupling in the optimal time-varying coupling is shown in the right column of Fig. 4.

In the next step, we explore how the optimal coupling can be applied to a concrete network. Hence, we apply the optimal time-varying coupling function to a ring network of 20 locally coupled Rössler systems, which is characterized by the network eigenvalues

$\lambda_i = 0, 0.0979, 0.0979, 0.3820, 0.3820, 0.8244, 0.8244, 1.3820, 1.3820, 2.0000, 2.0000, 2.6180, 2.6180, 3.1756, 3.1756, 3.6180, 3.6180, 3.9021, 3.9021, 4$. At first, strong coupling is applied until $t = 100$ to synchronize the systems. In our simulations, the three-variable coupling with $\sigma = 1.74$ has been used. However, any coupling can be used for the initial synchronization, and using three-variable coupling is just for achieving fast synchronization. By removing the coupling at $t = 100$, the systems remain synchronous for a finite transient time before diverging. The time series of the systems for this case are shown in Fig. 5(a). According to the MSF of the Rössler system (Fig. 1), the best constant coupling is $y \to y$ coupling, resulting in synchronization at $K = 0.159$. Therefore, the stable synchronization for the Rössler network is achieved at $\sigma = 1.622$ since $\lambda_2 = 0.098$. This critical coupling strength changes to $\sigma = 1.071$ for time-varying coupling whose MSF shows stable synchronization at $K = 0.105$. Consequently, we consider two cases by applying the best constant coupling and the optimal coupling.

In the first case, the best constant coupling ($y \to y$) is applied with the strength $\sigma = 1.23$ after $t = 100$. The time series of the systems are represented in Fig. 5(b). It can be observed that although the synchronization is maintained longer than in panel (a), the time series finally diverge and the systems become asynchronous. For the second case, to apply the optimal time-varying coupling, the final point of the synchronous trajectory is used for finding the optimal coupling function with $\sigma = 1.23$, and then, it is applied after $t = 100$. The results of the systems, which are shown in Fig. 5(c), indicate that the synchronization is preserved in this case.

IV. CONCLUSION

In this paper, the synchronization of oscillators coupled by a single variable was studied. It was proposed that a time-varying coupling function found by proper selection of the coupling-variable can lead to enhanced synchronization. According to the master stability function, the stability of synchronization depends on the maximum Lyapunov exponent of the linearized variational equations. Here, it was proposed that optimum time-varying coupling functions can be obtained by considering the local Lyapunov exponents. Short time intervals were considered, and the local Lyapunov exponents of all possible single-variable couplings were computed. In each time interval, the coupling that resulted in the minimum local Lyapunov exponent was selected. Finally, a time-varying coupling function was obtained for which the synchronization was more robust and could be achieved for smaller coupling parameters. We applied our method to networks of Rössler, Chen, and Chua systems. The results showed that the master stability function vs coupling parameter for time-varying coupling lies below that for any constant coupling, and thus, synchronization is enhanced. The optimal time-varying coupling function was also applied to a ring network of Rössler systems. It was shown that using the optimal coupling, the systems remain synchronous with coupling strength lower than the best constant single-variable coupling. It should be noted that the proposed optimal coupling can be applied to networks with any structure, including scale-free and small-world networks.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Zahra Dayani: Investigation (equal); Software (equal); Writing – original draft (equal). **Fatemeh Parastesh:** Investigation (equal); Software (equal); Writing – original draft (equal). **Fahimeh Nazarimehr:** Formal analysis (equal); Methodology (equal); Validation (equal); Visualization (equal); Writing – original draft (equal). **Karthikeyan Rajagopal:** Formal analysis (equal); Methodology (equal); Validation (equal); Visualization (equal); Writing – original draft (equal). **Sajad Jafari:** Conceptualization (equal); Methodology (equal); Writing – review & editing (equal). **Eckehard Schöll:** Conceptualization (equal); Supervision (equal); Writing – review & editing (equal). **Jürgen Kurths:** Conceptualization (equal); Supervision (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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