Alberti, T., Faranda, D., Lucarini, V., Donner, R. V., Dubrulle, B., Daviaud, F. (2023): Scale dependence of fractal dimension in deterministic and stochastic Lorenz-63 systems. - Chaos, 33, 2, 023144.

DOI: https://doi.org/10.1063/5.0106053

Chaos
An Interdisciplinary Journal of Nonlinear Science

RESEARCH ARTICLE | FEBRUARY 27 2023

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Special Collection: Theory-informed and Data-driven Approaches to Advance Climate Sciences

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Chaos 33, 023144 (2023)
https://doi.org/10.1063/5.0106053

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Cite as: Chaos **33**, 023144 (2023); doi: 10.1063/5.0106053
Submitted: 27 June 2022 · Accepted: 1 February 2023 ·
Published Online: 27 February 2023

View Online Export Citation CrossMark

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Note: This article is part of the Focus Issue, Theory-informed and Data-driven Approaches to Advance Climate Sciences.
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ABSTRACT

Many natural systems show emergent phenomena at different scales, leading to scaling regimes with signatures of deterministic chaos at large scales and an apparently random behavior at small scales. These features are usually investigated quantitatively by studying the properties of the underlying attractor, the compact object asymptotically hosting the trajectories of the system with their invariant density in the phase space. This multi-scale nature of natural systems makes it practically impossible to get a clear picture of the attracting set. Indeed, it spans over a wide range of spatial scales and may even change in time due to non-stationary forcing. Here, we combine an adaptive decomposition method with extreme value theory to study the properties of the instantaneous scale-dependent dimension, which has been recently introduced to characterize such temporal and spatial scale-dependent attractors in turbulence and astrophysics. To provide a quantitative analysis of the properties of this metric, we test it on the well-known low-dimensional deterministic Lorenz-63 system perturbed with additive or multiplicative noise. We demonstrate that the properties of the invariant set depend on the scale we are focusing on and that the scale-dependent dimensions can discriminate between additive and multiplicative noise despite the fact that the two cases have exactly the same stationary invariant measure at large scales. The proposed formalism can be generally helpful to investigate the role of multi-scale fluctuations within complex systems, allowing us to deal with the problem of characterizing the role of stochastic fluctuations across a wide range of physical systems.

The collective dynamics of natural systems is the result of the dynamics of their individual components, often operating on multiple spatiotemporal scales and sometimes related to intrinsic and extrinsic factors. These multiple components reflect in scaling laws, unpredictable vs deterministic behavior, bifurcations between different regimes, and basin of attractions. Here, we propose a novel concept of fractal dimension in deterministic and stochastic Lorenz-63 systems to provide a more complete characterization of the geometric features of attractors at different scales.

I. INTRODUCTION

Since their first description by Lorenz in 1963,[1] the existence and properties of strange attractors have been frequently discussed in the context of such diverse fields as the atmosphere,[2] climate,[3,4] biology,[5] and ecology,[6] to mention only a few examples. The concept of strange attractors is strictly related to that of dissipative dynamical systems with sensitive dependence on the initial conditions. Being revolutionary at the time of its invention, it has been attracting much attention, especially in the context of developing measures to quantify the geometric and dynamical properties of attractors[7] and in revising some earlier concepts on the forecast horizon of physical systems.[8] A one-parametric family of measures, the so-called generalized fractal dimensions D_q, has been proposed based on a coarse-grained invariant measure linking the geometric properties of the phase-space trajectories to the statistics of the dynamical scaling properties.[9] These measures provided new insights not only in the field of dynamical system theory (where they have been developed)[10] but also into different more applied fields, such as fluid and magneto-hydrodynamic turbulence[11,12] and others.[13]

One of the peculiar aspects of physical systems is their variability over a wide range of scales, arising from both intrinsic interactions between characteristic variability components in one or several variables and external forcings, differently affecting the specific properties of the whole system at different scales.[4] Recently, Alberti *et al.*[14] proposed a method to investigate how scale-dependency affects the global phase-space properties and their statistical characteristics. This method requires to first identify scale-dependent components contributing to the observed dynamics of a given system as a whole, which can be achieved by applying time series decomposition techniques, such as empirical mode decomposition (EMD). Subsequently, quantitative scale-specific measures, such as generalized fractal dimensions, are evaluated. The formalism resulting from the combination of those two approaches allows the introduction of *multi-scale measures* by computing the generalized fractal dimensions for each scale-specific component and partial sums thereof.[14] The suitability of this approach has been demonstrated for several dynamical systems of different complexity, synthetic noisy signals, and real-world time series data.[14]

For systems exhibiting a heterogeneous phase-space structure or even non-stationarity, it would, however, be useful to track the instantaneous number of degrees of freedoms, which are closely related to its associated recurrence characteristics.[15] Indeed, the spatial distribution of the instantaneous (i.e., local in phase-space) dimensions across the system's invariant set, as well as its geometric shape, provides us with more detailed information than established global (as in Hentschel and Procaccia[9]) and/or scale-dependent (as in Alberti *et al.*[14]) measures of complexity.

Accordingly, in this work, we thoroughly extend the existing formalism of multi-scale measures[14] to characterize the instantaneous scale-dependent properties of strange attractors by combining time series decomposition methods with concepts from extreme value theory that are related to the instantaneous number of degrees of freedom of the observed dynamics. We then show the utility of our approach for the case of the well-known low-dimensional deterministically-chaotic Lorenz-63 system and two stochastic versions thereof.[16] We indeed show that the new formalism, based on instantaneous scale-dependent dimensions, allows us to discern two properties that are inaccessible by previous global or scale-dependent analysis, namely, the existence of different scale-dependent source processes (as the presence of noise or a dominant scale) and the structural stability of fixed points.

II. METHODS

In this section, we start by introducing the decomposition procedure and the dynamical system metrics separately, before describing our proposed formalism. For a more general purpose, we assume to have a generic N-dimensional system, i.e., an $N-$dimensional phase-space, with $N > 1$. Thus, we describe our decomposition procedure in a general multivariate framework. For univariate data (i.e., $N = 1$), we may proceed in a largely analogous way.

A. Multivariate empirical mode decomposition (MEMD)

Considering an N-dimensional system described via a multivariate time series signal $\Theta_\mu(t) = [\Theta_1(t), \Theta_2(t), \ldots, \Theta_N(t)]^\dagger$ (with $\dagger$ indicating the transposition operator), the multivariate empirical mode decomposition (MEMD) decomposes the data into a finite number of multivariate oscillating patterns $\mathbf{C}_{\mu,k}(t)$, referred to as multivariate intrinsic mode functions (MIMFs), and a monotonic residue $\mathbf{R}_\mu(t)$ as

$$\Theta_\mu(t) = \sum_{k=1}^{n_k} \mathbf{C}_{\mu,k}(t) + \mathbf{R}_\mu(t). \tag{1}$$

The decomposition basis, formed by the set of functions $\mathbf{C}_{\mu,k}(t)$, is empirically derived via the so-called sifting process[17] modified for multivariate signals.[18] This sifting process consists of

1. identifying local extremes of $\Theta_\mu(t)$, i.e., where the N-variate derivative vanishes;
2. interpolating these points via cubic splines to derive the upper and lower envelopes $\mathbf{u}(t)$ and $\mathbf{l}(t)$, respectively;
3. deriving the mean envelope $\mathbf{m}(t)$ as $\mathbf{m}(t) = \frac{\mathbf{u}(t)+\mathbf{l}(t)}{2}$; and
4. evaluating the detail $\mathbf{h}(t) = \mathbf{s}(t) - \mathbf{m}(t)$.

These steps are iterated until the detail $\mathbf{h}(t)$ can be identified as a MIMF (also called a multivariate empirical mode);[18] i.e., it must satisfy two properties: it has the same number of local extremes and zeros (or both differing at most by one) and a zero-average mean envelope $\mathbf{m}(t)$.[17] The full sifting process stops when no more MIMFs $\mathbf{C}_{\mu,k}(t)$ can be filtered out from the data. Each $\mathbf{C}_{\mu,k}(t)$ represents a peculiar dynamical component intrinsic to the system that typically evolves on an average scale

$$\tau_k = \frac{1}{T}\int_0^T t' \langle \mathbf{C}_{\mu,k}(t') \rangle \, dt', \tag{2}$$

where T is the length of data and $\langle\cdot\rangle$ denotes an ensemble average over the N-dimensional space.[19] The MEMD allows us to interpret $\Theta_\mu(t)$ as a collection of scale-dependent multivariate fluctuations contributing to the collective properties of the whole system. Indeed, each MIMF can be seen as a representative of fluctuations at a typical scale that is the average of the instantaneous scales (i.e., the inverse instantaneous frequencies) derived from a given mode via the Hilbert transform.[19] The MEMD, due to its adaptive methodology, relieves some *a priori* mathematical constraints of fixed-basis decomposition methods and extracts a limited number of intrinsic components that can be visually inspected. Other widely used decomposition methods, such as Fourier or continuous wavelet analysis, commonly return a large number of components and/or need to project our data on a pre-defined decomposition basis. Moreover, at least classical Fourier transform based methods also require that our data satisfy a stationarity condition. In this regard, we do not question the appropriateness of the aforementioned more traditional conventional analysis techniques, but rather acknowledge that they (as well as any other approaches) have intrinsic limitations in what we can learn from their application.

B. Instantaneous dimension

Given the N-dimensional system described via the multivariate trajectory $\Theta_\mu(t)$, its dynamical properties can be investigated by combining the concept of recurrences in phase-space and extreme value theory.[20] For some (arbitrary) state of interest ζ in the associated phase-space, we first introduce the logarithmic return associated with each state on the trajectory (except for ζ itself) as

$$G(\Theta_\mu(t), \zeta) = -\log\left[\operatorname{dist}(\Theta_\mu(t), \zeta)\right], \tag{3}$$

where $\operatorname{dist}(\cdot)$ is a distance between two state vectors in phase-space, commonly the Euclidean one. By shortening the notation, we obtain a time series of logarithmic returns $g(t) = G(\Theta_\mu(t), \zeta)$ that takes larger values whenever $\Theta_\mu(t)$ is close to ζ. If we now define a threshold $s(q)$ as the qth empirical quantile of $g(t)$, we can introduce the exceedances $u(\zeta) \doteq \{t \,|\, g(t) > s(q)\}$, i.e., the recurrences to the neighborhood of the reference state in the context first introduced by Poincaré by exploiting a peak-over-threshold like concept as widely used in extreme value theory. According to the Freitas–Freitas–Todd theorem, the cumulative probability distribution $F(u, \zeta)$ then converges to the exponential member of the generalized Pareto distribution (GPD); i.e.,

$$F(u, \zeta) \simeq \exp\left[-\frac{u(\zeta)}{\varsigma(\zeta)}\right]. \tag{4}$$

The GPD parameter ς depends on the dynamical state ζ and can be used to introduce the concept of an instantaneous dimension d at the point in time where ζ is attained, which is simply defined as $d(\zeta) = \varsigma(\zeta)^{-1}$. Although it could merely be associated with a fitting parameter, it has a clear physical meaning: d is a proxy of the active number of degrees of freedom around each state ζ in the phase-space. Note, however, that from a practical perspective, this instantaneous dimension needs to be considered relative to the set of time series values available, and the interpretation of its values may be affected by nonstationarity or non-representative sampling of the presumed attractor at finer spatial scales.

C. Instantaneous scale-dependent dimension

The instantaneous dimension d introduced above provides a local (in terms of phase-space) picture of the properties of phase-space trajectories, i.e., allows us to obtain information for each sampled point contributing to the global structure of the attractor under study. Nevertheless, multi-scale systems could have a scale-dependent phase-space structure[14] such that we can distinguish between features that emerge at different scales. To provide a scale-dependent instantaneous view of a given system, we have to combine a decomposition method, such as the MEMD, and the extreme value theory applied to inter-state distances in phase space.

Given again an N-dimensional system described via $\Theta_\mu(t)$ with a multi-scale nature, i.e., being characterized by processes occurring over a wide range of scales, we can write

$$\Theta_\mu(t) = \langle\Theta_\mu(t)\rangle + \sum_\tau \delta\Theta_\mu^{(\tau)}(t), \tag{5}$$

where $\langle\Theta_\mu(t)\rangle$ is a steady-state time-average value and $\delta\Theta_\mu^{(\tau)}(t)$ is a component of the system operating at a mean scale τ. It is easy to note the analogy between Eqs. (1) and (5) via the correspondence $\mathbf{C}_{\mu,k}(t) \leftrightarrow \delta\Theta_\mu^{(\tau)}(t)$ and $\mathbf{R}_\mu(t) \leftrightarrow \langle\Theta_\mu(t)\rangle$. This means that for each scale τ, we can identify the corresponding invariant set $\mathbb{M}_\tau$ as the manifold obtained via the partial sums of MIMFs with scales $\tau_\star < \tau$; i.e.,

$$\Theta_\mu^\tau(t) = \sum_{k^\star=1}^{k} \mathbf{C}_{\mu,k^\star}(t). \tag{6}$$

Then, for each scale $\tau \in [\tau_1, \tau_{n_k}]$, i.e., for each $k \in [1, n_k]$, given a trajectory $\Theta_\mu^\tau(t)$ and a state of interest ζ_τ, the cumulative probability of logarithmic returns in the neighborhood of ζ_τ follows a GPD as

$$F(u_\tau, \zeta_\tau) \simeq \exp\left[-\frac{u_\tau(\zeta_\tau)}{\varsigma_\tau(\zeta_\tau)}\right]. \tag{7}$$

Thus, we can introduce a quantity $D(t, \tau) = \varsigma_\tau(\zeta_\tau)^{-1}$, representing the number of active degrees of freedom of fluctuations up to a maximum scale of τ around each state ζ_τ. In this way, we exploit the properties of MEMD in deriving scale-dependent components embedded into a given system and the instantaneous (in terms of time) properties of the extreme value theory based metric to derive the instantaneous scale-dependent metric $D(t, \tau)$.

In summary, our procedure consists of the following steps:

1. extract intrinsic components $\mathbf{C}_{\mu,k}(t)$ and their mean scales τ_k from $\Theta_\mu(t)$ by using the MEMD;
2. evaluate partial sums of Eq. (1) at different scales

$$\Theta_\mu^\tau(t) = \sum_{k^\star=1}^{k} \mathbf{C}_{\mu,k^\star}(t), \tag{8}$$

   with $k^* = 1, \ldots, n_k$ (by construction, MIMFs are ordered with increasing scales, i.e., $\tau_{k'} < \tau_{k''}$ if $k' < k''$); and
3. for each scale τ_k (i.e., for each k), evaluate $D(t, \tau_k)$.

Our procedure is, by construction, complete since when $k \to n_k$, then $D(t, \tau_k) = d(t)$, with $d(t)$ being the instantaneous fractal dimension of the full system.[2,15,20]

In the remainder of this work, we will discuss some examples to highlight the potential of our framework to disentangle distinct dynamical components of a different origin in a multi-scale complex system.

III. THE LORENZ-63 MODEL AND ITS STOCHASTIC VERSIONS

The Lorenz-63 system,[1] originally developed as a simplified model for atmospheric convection, is one of the most famous and widely studied paradigmatic dissipative–chaotic dynamical systems,[21] which can be written as

$$dx = s\left(-x + y\right) dt, \tag{9}$$

$$dy = \left(rx - y - xz\right) dt, \tag{10}$$

$$dz = \left(xy - bz\right) dt, \tag{11}$$

with the parameters $\left(s, r, b\right)$ related to the Prandtl number, the Rayleigh number, and the geometry of the atmospheric convective layer. With the canonical set of parameters $\left(s, r, b\right) = (10, 28, 8/3)$, the system admits chaotic behavior with all initial points (except for a set of measure zero) tending toward an invariant set with the fractal structure, usually termed the Lorenz attractor. It is a strange attractor whose Hausdorff dimension (and all its generalizations D_q[9]) takes a value of 2.05 ± 0.02.[7,9,22]

A simple way to investigate the role of hidden fast dynamical components is to couple deterministic equations to "noise" mimicking the action of unknown fast variables. This can also be easily done for the Lorenz-63 system by rewriting the original system in terms of a set of coupled stochastic differential equations as

$$dx = s\left(-x + y\right) dt + \sigma\, dW_t, \tag{12}$$

$$dy = \left(rx - y - xz\right) dt + \sigma\, dW_t, \tag{13}$$

$$dz = \left(xy - bz\right) dt + \sigma\, dW_t. \tag{14}$$

In nonlinear deterministic systems, such additive noise can lead to non-trivial effects,[23] including transitions between coexisting states or attractors, shifting bifurcations, or acting as external forcing to the intrinsic variability of the system,[11,24] also observed for the Lorenz-63 system.[16]

More recently, another stochastic version of the Lorenz-63 system has been proposed by Chekroun *et al.*,[16] considering a linearly multiplicative noise term to the original system as

$$dx = s\left(-x + y\right) dt + \sigma\, x\, dW_t, \tag{15}$$

$$dy = \left(rx - y - xz\right) dt + \sigma\, y\, dW_t, \tag{16}$$

$$dz = \left(xy - bz\right) dt + \sigma\, z\, dW_t. \tag{17}$$

This system provided a first example for the existence of random attractors, extending the concept of a strange attractor, still supporting nontrivial sample measures from deterministic to stochastic dynamics[1,16] that have been shown to be random Sinaï–Ruelle–Bowen measures.[25] Note that in the weak-noise limit, response theory allows one to compute explicitly the change in the expectation value of the measurable observables when perturbing an underlying chaotic dynamics with stochastic terms of rather general nature.[26]

In the following, we apply our formalism to the three different versions of the Lorenz-63 system described above. In the case of the stochastic models featuring multiplicative noise, we use the Itô convention for the stochastic integration, and dW_t is a Wiener process obtained by sampling at each time step a random variable with the same Gaussian density $[W_t \sim \mathcal{N}(0, t)]$ and intensity σ. The numerical simulation of Eqs. (9)–(17) is obtained by using the Euler–Maruyama method with a time resolution $dt = 5 \times 10^{-3}$ over $N = 10^7$ time steps, using the classical set of parameters $\left(s, r, b\right) = (10, 28, 8/3)$ and $\sigma = 0.4$ as in Chekroun *et al.*[16]

IV. RESULTS

A. Full system attractor

Figures 1–3 report the trajectories (left panels) of the three different Lorenz-63 systems (deterministic, Fig. 1; additive noise, Fig. 2; multiplicative noise, Fig. 3) and their corresponding attractors in the 3D phase-space (right panels). As expected, a breakdown of the symmetric shape of the Lorenz attractor is observed when the classical Lorenz-63 system is subject to either additive or multiplicative noise. Furthermore, intermittency appears to be reduced, thus moving from a deterministic strange attractor toward a random stochastic attractor.[16] Nevertheless, by only looking at the full system attractor, we are not able to identify any significant difference in the geometric shape between the additive and multiplicative model. Furthermore, both random attractors are characterized by the same dimension, equal to the full dimension of the phase-space, due to the corresponding property of the invariant measure of an elliptic diffusion process.

B. Average dimensions of scale-dependent attractors

To further inspect and characterize the role of the noise vs the deterministic dynamics of the Lorenz system, we apply our formalism to derive $D(t, \tau)$ for the three different systems. We first decompose every multivariate trajectory $\Theta_\mu(t)$ via the MEMD through which we obtained a set of 15, 20, and 24 MIMFs, respectively, whose ranges of timescales are $\tau \in [1.64, 2.5 \times 10^5]$, $\tau \in [9.7 \times 10^{-2}, 2.5 \times 10^5]$, and $\tau \in [8.9 \times 10^{-2}, 2.5 \times 10^5]$, respectively. Then, using Eqs. (6) and (7), we derive the instantaneous scale-dependent metric $D(\tau, t)$ for the three systems.

As a first step, we inspect the behavior of the average instantaneous scale-dependent dimension $\langle D(\tau, t)\rangle_t$ as a function of the scale τ as reported in Fig. 4. This is equivalent to the method proposed by Alberti *et al.*[14] where local (in terms of timescale) and time-independent (i.e., averaged) multi-scale measures have been introduced.

Our corresponding analysis evidences the absence of timescales $\tau < 1.64$ for the deterministic Lorenz-63 system as opposed to its stochastic versions. This is clearly a reflection of the absence of stochastic terms in the classical deterministic system, which are responsible for the very fast fluctuations in the stochastic cases.

Furthermore, we generally observe larger average scale-dependent dimensions for the multiplicative noise case than for the deterministic Lorenz-63 system and the additive noise model. This reflects the effect of the stochastic term on the dynamical

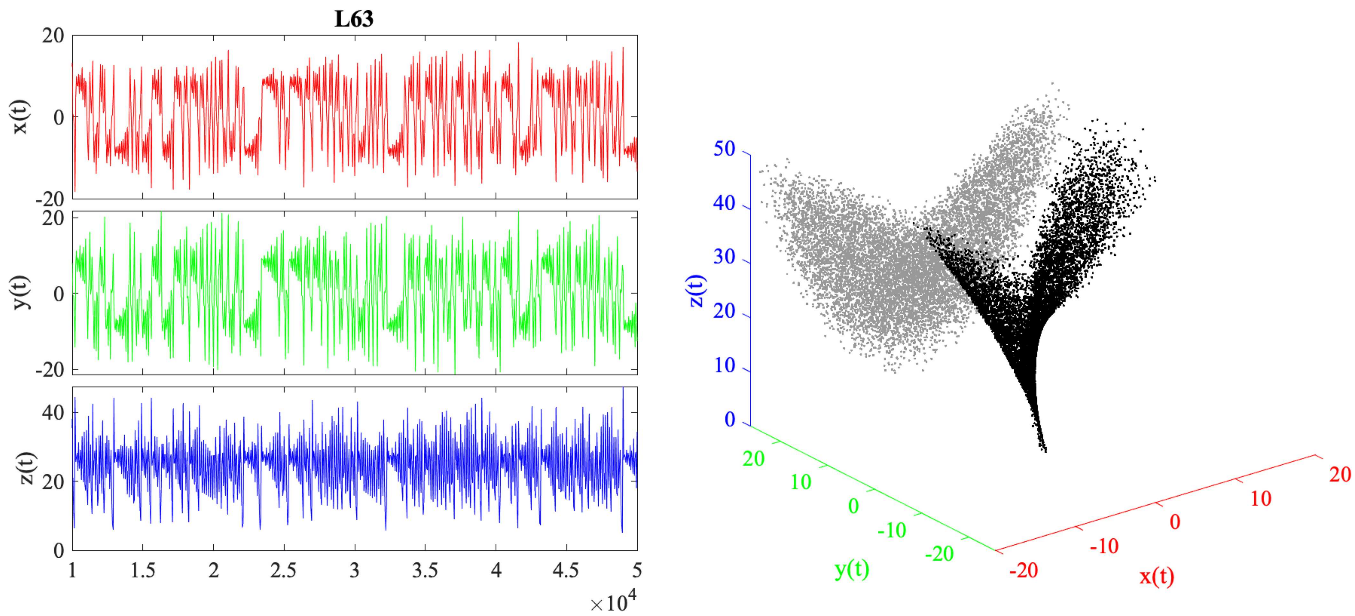

FIG. 1. (Left) Zoom of the trajectory components of the deterministic Lorenz-63 system as in Eqs. (9)–(11) (L63). (Right) Corresponding attractor in the 3D phase-space (black points) and its projection in the x–z plane (gray points).

features of the Lorenz-63 system: it does not only act at short scales, exciting variability at additional scales with respect to the classical Lorenz-63 system, but also affects the attractor geometry and, hence, the time-averaged number of active degrees of freedom as reflected by the scale-dependent fractal dimension metric at larger timescales. The latter property can be linked to the fact that the Lorenz-63 system with non-degenerate noise has an invariant measure that is absolutely continuous with respect to Lebesgue

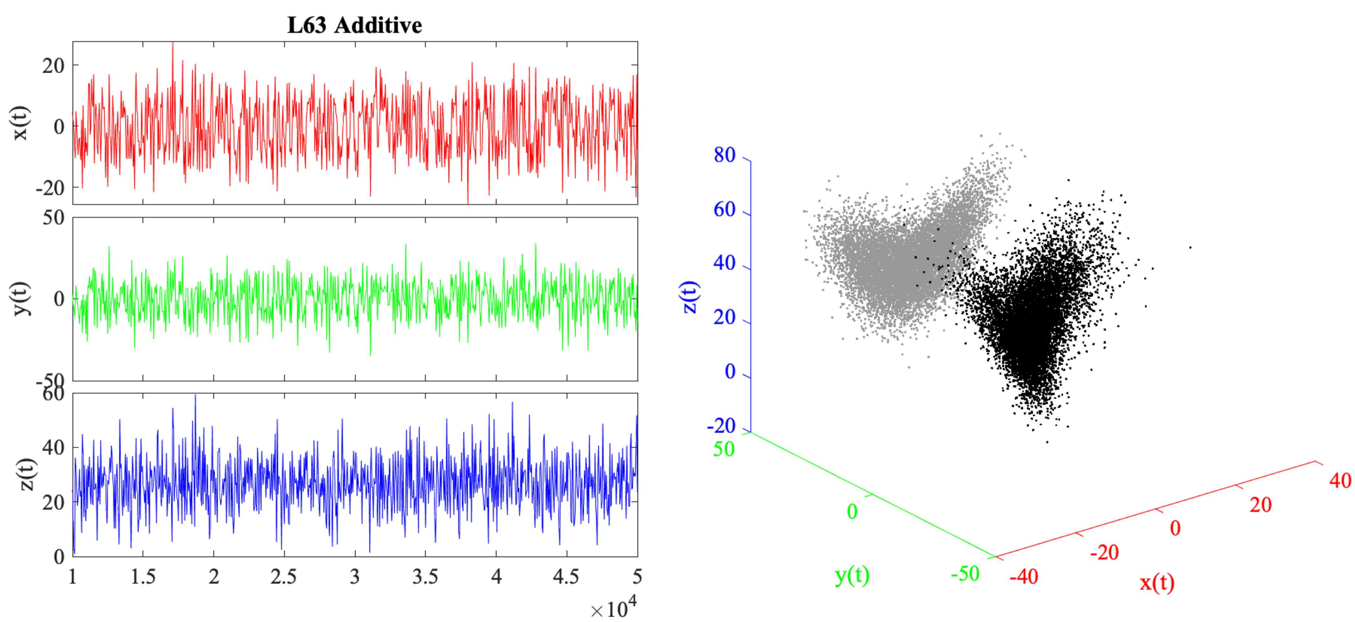

FIG. 2. (Left) Zoom of the trajectory components of the additive noise model as in Eqs. (12)–(14) (L63 additive). (Right) Corresponding attractor in the 3D phase-space (black points) and its projection in the x–z plane (gray points). The stochastic noise term has an amplitude $\sigma = 0.4$.[16]

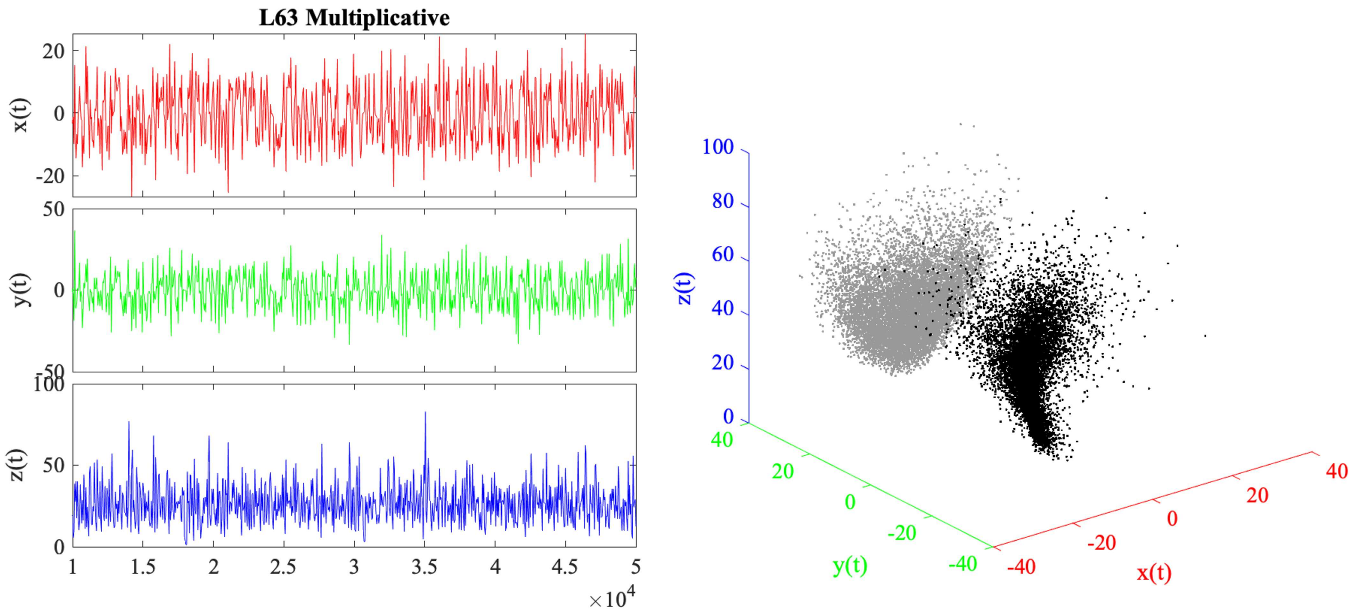

FIG. 3. (Left) Zoom of the trajectory components of the multiplicative noise model as in Eqs. (15)–(17) (L63 multiplicative). (Right) Corresponding attractor in the 3D phase-space (black points) and its projection in the x–z plane (gray points). The stochastic noise term has an amplitude $\sigma = 0.4$.[16]

such that when introducing a stochastic term, the dimension must converge to 3, as observed at large timescales for both stochastic models.

The most interesting feature emerging for the average dimensions is that the largest value $D(t, \tau)$ for both the additive and multiplicative case is obtained for τ of the order of the fundamental period (i.e., the Lyapunov timescale $\tau_L \approx 1.12$ time units) of the dominating unstable periodic orbit of the deterministic system;[27,28] see Ref. 29 for a discussion of how unstable periodic orbits are responsible for resonant behavior in forced systems and Ref. 30 for evidence of the resonant response of the Lorenz-63 system.

Another interesting feature is the quantitative difference of $\langle D(\tau, t)\rangle$ in the range of scales dominated by the stochastic contribution (below τ_L). While for the additive model the average dimensions converge, as expected, to 3, the dimensions are larger than 3 for the multiplicative case at the same timescales, thus suggesting that the dynamics at these scales behave as a forcing-like contribution. Conversely, by looking at the full system attractor, i.e., when considering the whole timescales, we are not able to identify any difference in the dimensions between the additive and multiplicative model. Indeed, as expected, the average dimensions tend to saturate to those expected for the full dynamics when $\tau \to \tau_{N_k}$, being $\langle D(\tau, t)\rangle = 2.05 \pm 0.02$ for the deterministic Lorenz-63 system and $\langle D(\tau, t)\rangle = 2.98 \pm 0.04$ for its stochastic versions, because the invariant measure of an elliptic diffusion process has full dimension. The results obtained for the deterministic Lorenz-63 system are in agreement with previous findings by Alberti *et al.*,[14] while our corresponding findings on the stochastic models have been reported here for the first time.

C. Instantaneous scale-dependent dimensions

As a second step of our analysis, which is also the main novelty introduced in this work, we investigate the behavior of the instantaneous scale-dependent dimension $D(t, \tau)$ for the three different systems as reported in Fig. 5.

The deterministic Lorenz-63 system is characterized by instantaneous dimension values close to $D_F \simeq 2.05$ at timescales larger than 10^2, as expected due to the monofractal nature of the system with all generalized fractal dimensions D_q taking the same value for the full system. Conversely, larger values are found for the range $\tau \in [10^0, 10^2]$, the reason for which will be discussed below.

When a stochastic term is considered, we observe values of $D(t, \tau) \gtrsim 3$ at short timescales, extending toward larger scales when localized (in time) intermittent bursts in the trajectory take place. Typically, dimensions larger than 3 imply the presence of external forcing components, increasing the number of active degrees of freedom. By further inspecting the behavior of the trajectory in the phase-space at large scales (see Fig. 6), this excess over the topological dimension of the phase-space appears to be related to situations associated with approaches of the unstable fixed points in the centers of each of the two lobes and subsequent fast escapes from the neighborhoods of those points along their unstable manifolds. This indicates that the increase in the number of active degrees of freedom at short timescales, not observed for the deterministic model, is related to the stochastic component. Indeed, the imposed noise term acts as additional forcing to the autonomous dynamics. This means that the noise introduces additional degrees of freedom in the dynamics because it adds energy to the system: the attractor can deform through scales by increasing/decreasing its dimensions

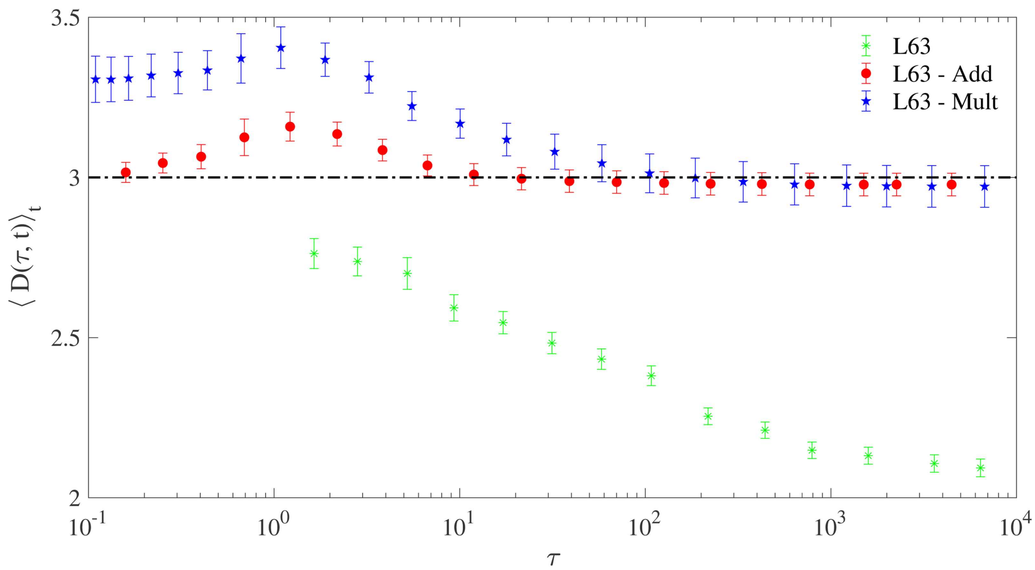

FIG. 4. Average instantaneous scale-dependent dimension $\langle D(\tau, t)\rangle_t$ as a function of the scale τ. Green asterisks refer to the Lorenz "63 system, red circles to the additive noise model, and blue stars to the multiplicative noise case. Error bars are obtained as the standard deviations of $D(t, \tau)$ along the respective trajectory.

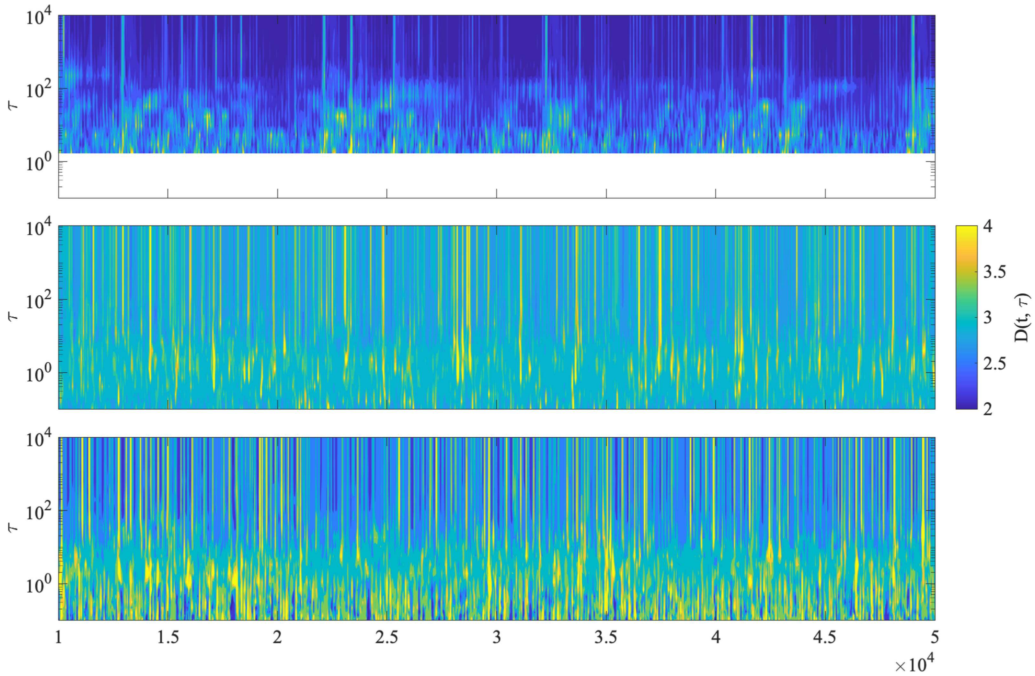

FIG. 5. Behavior of the instantaneous scale-dependent dimension $D(t, \tau)$ for the deterministic Lorenz-63 system (top), its version with additive noise (middle), and the multiplicative noise case (bottom). The colormap for $D(t, \tau)$ has been saturated between 2 and 4 for better visualization. In all three cases, an excerpt comprising 40 000 time units is shown.

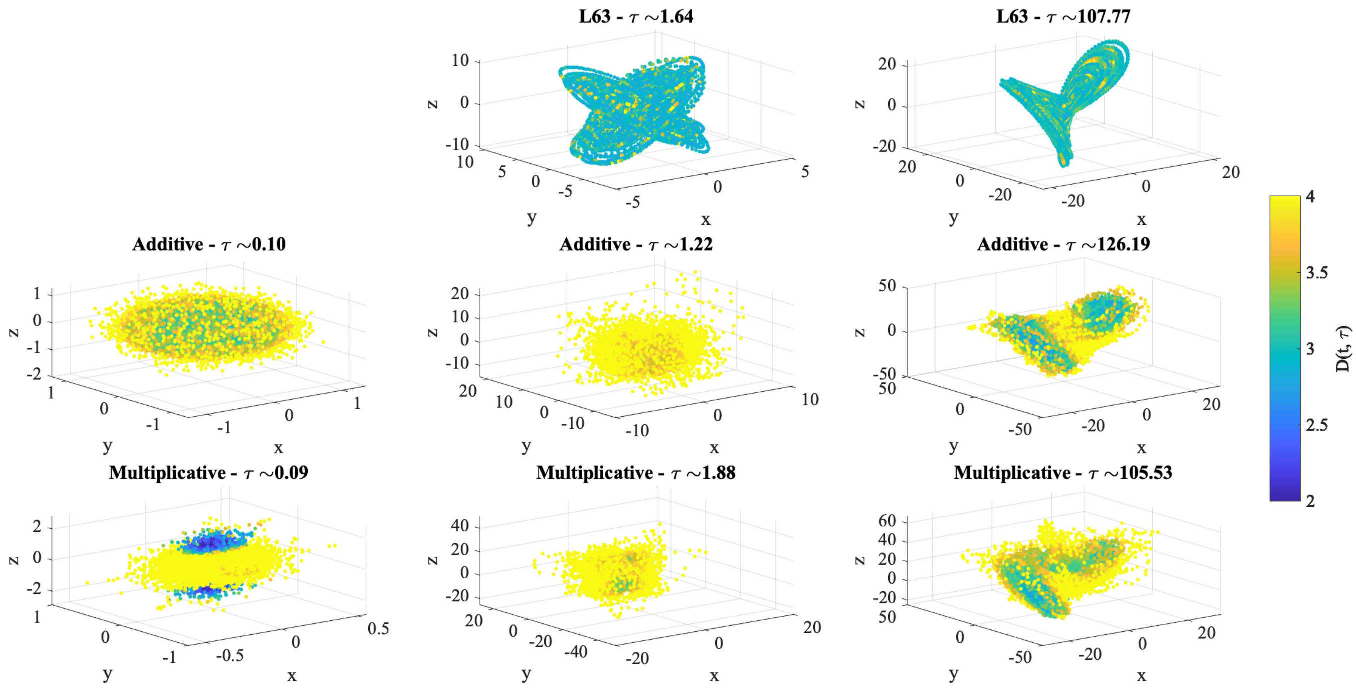

FIG. 6. Three views of the Lorenz attractor at different timescales color-coded with respect to the instantaneous dimensions: (top) deterministic system, (middle) additive noise, and (bottom) multiplicative noise.

depending on the instantaneous concurrent effect between the noise forcing term and the intrinsic dynamics of the Lorenz-63 system. The main differences between the two stochastic versions emerge at short timescales ($\tau < 10^0$), where larger dimensions are found for the multiplicative noise case as compared to the additive one. This could be explained by invoking the fact that in the multiplicative case, the amplitude of the stochastic term depends on the state variables of the system.

D. A scale-dependent instantaneous view of the attractor

As a final step and to better highlight the scale-dependent instantaneous properties of the attractor, Fig. 6 reports three views of the attractor at different timescales color-coded with respect to the instantaneous dimensions. The shapes of the different scale-dependent attractors are obtained by summing up empirical modes in a certain range of scales as in Eq. (8). Due to the associated properties of MIMFs, they have a zero-average envelope; thus, they fluctuate around zero such that the attractors occupy only a small region (especially, at short scales) of the phase-space of scale-dependent fluctuations.

At large timescales (right panels in Fig. 6), roughly corresponding to 100 times the Lyapunov time of the deterministic Lorenz-63 system at the considered parameter values, the geometric shapes and the spatial distribution of dimensions across both stochastic attractors are qualitatively similar and clearly distinct from the chaotic attractor of the deterministic model. This means that we can visually distinguish between the chaotic and the two stochastic attractors, while a clear distinction cannot be made qualitatively and quantitatively between the two random invariant sets.

By further inspecting the spatial distributions of instantaneous dimensions of all three attractors, we clearly observe that larger dimensions are found at the edges of the attractors and close to the origin as compared to lower dimensions observed within the two lobes. While this feature has also been previously highlighted for the deterministic chaotic attractor (see, e.g., Faranda *et al.*[31]), it is the first time that the spatial distribution of dimensions is inspected for both random attractors. In particular, our analysis reveals that forcing-like mechanisms leading to $D(\tau, t) > 3$ are operating at the edges of the attractor and close to the unstable fixed point at the origin, reflecting the repelling nature of the fixed points of the Lorenz-63 system. However, the observation of $D(\tau, t) > 3$ suggests that the number of active degrees of freedom near the origin is increased with respect to the deterministic model, likely related to the action of the noise term.

To further investigate the last point on the role of the stochastic fluctuations in increasing the number of degrees of freedom near the origin, we investigate the spatial distribution of dimensions across the trajectory at short timescales, below the Lyapunov time (left panels in Fig. 6). Clearly, we do not have any dynamical component below the Lyapunov time for the deterministic Lorenz-63 system, indicating that in both noisy systems, variability at those fast timescales is intimately related to the stochastic forcings.

However, completely different spatial distributions of dimensions across the trajectory are observed between the additive and multiplicative case. While the former has a more homogeneous spatial distribution of dimensions with the most probable value close to 3 and small fluctuations around it, the latter is characterized by a saddle point-like dynamics $D(\tau, t) > 3$ in a ring-like configuration lying in the x–y plane and $D(\tau, t) < 3$ elongated in the z direction. This is due to the different structure of the noise terms, being a "pure" noise term in the additive case, reflecting into $D(\tau, t) = 3 + \varepsilon$, with $\varepsilon \ll 1$, while acting as "forcing" for the multiplicative one, providing $D(\tau, t)$ values larger than the system's dimension.[15,20]

When approaching the Lyapunov scale τ_L (middle panels in Fig. 6), a different spatial distribution of the dimensions is again observed, together with a different coverage of the available phase-space when comparing the chaotic attractor with the two stochastic ones. The latter are characterized by regions with a low dimension surrounded by higher-dimensional ones, markedly differing from the deterministic Lorenz-63 system. We hypothesize that the regions with low instantaneous dimensions could indicate the location of weakly repulsive low-period unstable periodic orbits.[32,33]

Overall, our results indicate, for both the deterministic and the two stochastic models, a clear different spatial distribution of the dimensions across the phase-space at large timescales. In particular, the two stochastic attractors are characterized by a region with dimensions larger than the topological dimension of the system close to the origin $O = (0, 0, 0)$, suggesting the existence of forcing-like mechanisms altering the structure of this fixed point of the deterministic model. Since the stochastic term mainly operates at short timescales (below the Lyapunov scale τ_L), the source of this difference must be searched in this range of scales whose corresponding attractors for the additive and multiplicative models are structurally different, both in terms of the geometric shape and in the spatial distribution of the dimension values. In this regard, the stochastic term is able to change the stability of the origin, revealing a new structure of attractors whose properties (i.e., fractal dimensions) evolve in time and across scales. This difference disappears when reaching larger and larger timescales due to the existence of an invariant measure for the Lorenz-63 system that is absolutely continuous with respect to Lebesgue such that when introducing a stochastic term, the dimension must converge to 3, as observed at large timescales.

Finally, it is important to underline that in the deterministic Lorenz-63 system with standard parameters [i.e., $(s, r, b) = (10, 28, 8/3)$], the origin O is an unstable saddle point whose structure is preserved when considering a multiplicative stochastic term. Conversely, this nature seems to be modified when considering additive noise, altering the structural stability of the unstable point toward different nature. However, a linear stability analysis of the stochastic models is beyond the scope of the present work and is left for a future devoted study.

V. CONCLUSIONS

We have presented a formalism to study the behavior of chaotic or stochastic attractors as a function of the timescale, indicating that when considering different timescales, the concept of a single universal attractor should be revised. Specifically, using the famous Lorenz-63 system in its standard deterministic as well as two stochastically forced versions, we have demonstrated that the attractor of this system is scale dependent.

To reach this conclusion, we have extended an approach recently introduced by Alberti *et al.*[14] to investigate the instantaneous scale-dependent properties of attractors by combining concepts from time series decomposition methods and extreme value theory applied to recurrences in phase space. More specifically, we have used the multivariate empirical mode decomposition (MEMD) to derive intrinsic components of a given system at different timescales. Based on this decomposition, we have estimated the instantaneous scale-dependent dimensions of the system's attractor at different scales. We have shown that a new structure of attractors, whose properties evolve in time, space, and scale, is discovered by looking for fixed points and following their evolution from a small to large scale and vice versa. Thus, the geometric structure of the attractor is gradually deformed and depends on the scale at which we are investigating the respective system.

The main novelty introduced in this study is a powerful method to identify the existence of processes of a different origin by looking at the spatial distribution of fractal dimensions across the full phase-space trajectories at different timescales. Concerning the systems considered in this work, our formalism allowed us to clearly distinguish between a purely noise-like contribution at short timescales for the additive noise model, being characterized by an ergodic coverage of the available phase-space with dimensions fluctuating around 3 (as expected), as compared to a more forcing-like contribution for the multiplicative noise model at the same timescales, where the dimensions are larger than 3 and differently distributed across the attractor. Conversely, by looking at the full system attractor, i.e., when considering the whole range of contributing timescales, we are not able to identify any clear difference, both in the attractor shape and in the spatial pattern of the instantaneous (i.e., local in time and phase-space) dimensions, between the additive and multiplicative model since the invariant measure of an elliptic diffusion process has a full dimension. Thus, our method allows us to evidence where the difference between the two stochastic models resides only by looking at the spatial distribution (see Fig. 6) of instantaneous fractal dimensions at different timescales.

Our formalism can be easily modified by using any alternative time series decomposition technique (such as wavelet decomposition, singular spectrum analysis, or others). Our choice of the MEMD has been motivated by its empirical and adaptive nature, reducing *a priori* constraints and possible artifacts of fixed-frequency/fixed-basis decomposition methods. Furthermore, the instantaneous nature (i.e., time-dependency) of the intrinsic components derived via the MEMD allows us to perform a more detailed investigation of the dynamical evolution (in time) of a system variable, better suited for evaluating instantaneous dynamical system metrics (as the dimension) than fixed-basis methods as Fourier transforms.

We are confident that the proposed formalism provides a novel way to investigate the underlying geometric (fractal) properties of physical systems at different scales during their time evolution. The concept of a scale-dependent attractor could tackle the problem of defining a more useful concept for the analysis of multiscale systems, such as in the case of the climate or for turbulence, which has largely remained unsolved despite numerous efforts reported in the last four decades. In a companion paper,[34] where our formalism has been applied to laboratory experiments on fluids, we observe the emergence of an intrinsic timescale, solely determined by nonlinear interactions, controlling the geometric and topological properties of phase-space trajectories.

In this first study, we focused only on the geometric properties of attractors in order to show that the universal concept of an attractor can be insufficient for fully describing multiscale systems in the presence or absence of noise. The counterpart of our geometric view of each point in phase-space is the instantaneous, i.e., time behavior, of the scale-dependent dimension. Indeed, this interesting aspect can be used for further studying some crucial aspects of physical systems, e.g., bifurcations, tipping points, small- vs large-scale forcing, and/or driving mechanisms.[19] The corresponding prospects call for further studies to investigate these aspects in more detail, which is beyond the scope of the present paper and will be the subject of future work.

ACKNOWLEDGMENTS

We wish to acknowledge the support by ANR TILT Grant Agreement No. ANR-20-CE30-0035. V.L. acknowledges the support received from the Horizon 2020 project TiPES (Grant No. 820970) and from the EPSRC project EP/T018178/1. R.V.D. has received funding by the German Federal Ministry for Education and Research via the JPI Climate/JPI Oceans project ROADMAP (Grant No. 01LP2002B). The authors thank two anonymous reviewers for fruitful suggestions.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

T. Alberti: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **D. Faranda:** Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Software (equal); Writing – review & editing (equal). **V. Lucarini:** Conceptualization (equal); Investigation (equal); Methodology (equal); Validation (equal); Writing – original draft (equal); Writing – review & editing (equal). **R. V. Donner:** Investigation (equal); Methodology (equal); Supervision (equal); Validation (equal); Writing – review & editing (equal). **B. Dubrulle:** Investigation (equal); Validation (equal); Writing – review & editing (equal). **F. Daviaud:** Supervision (equal); Validation (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

[1] E. N. Lorenz, "Deterministic nonperiodic flow," J. Atmos. Sci. **20**, 130–148 (1963).

[2] D. Faranda, G. Messori, and S. Vannitsem, "Attractor dimension of time-averaged climate observables: Insights from a low-order ocean-atmosphere model," Tellus A: Dyn. Meteorol. Oceanogr. **71**, 1554413 (2019).

[3] C. Nicolis and G. Nicolis, "Is there a climatic attractor?," Nature **311**, 529–532 (1984).

[4] M. Ghil and V. Lucarini, "The physics of climate variability and climate change," Rev. Mod. Phys. **92**, 035002 (2020).

[5] S. Nikolov, O. Wolkenhauer, and J. Vera, "Tumors as chaotic attractors," Mol. BioSyst. **10**, 172–179 (2014).

[6] W. M. Schaffer and M. Kot, "Do strange attractors govern ecological systems?," BioScience **35**, 342–350 (1985).

[7] P. Grassberger and I. Procaccia, "Characterization of strange attractors," Phys. Rev. Lett. **50**, 346–349 (1983).

[8] A. N. Kolmogorov, "Entropy per unit time as a metric invariant of automorphism," Dokl. Russ. Acad. Sci. **124**, 754–755 (1959).

[9] H. G. E. Hentschel and I. Procaccia, "The infinite number of generalized dimensions of fractals and strange attractors," Physica D **8**, 435–444 (1983).

[10] J. L. Kaplan and J. A. Yorke, "Chaotic behavior of multidimensional difference equations," in *Functional Differential Equations and Approximation of Fixed Points*, edited by H.-O. Peitgen and H.-O. Walther (Springer, Berlin, 1979), pp. 204–227.

[11] R. Benzi, G. Paladin, A. Vulpiani, and G. Parisi, "On the multifractal nature of fully developed turbulence and chaotic systems," J. Phys. A **17**, 3521–3531 (1984).

[12] W. M. Macek, R. Bruno, and G. Consolini, "Generalized dimensions for fluctuations in the solar wind," Phys. Rev. E **72**, 017202 (2005).

[13] M. Cencini and F. Ginelli, "Lyapunov analysis: From dynamical systems theory to applications," J. Phys. A **46**, 250301 (2013).

[14] T. Alberti, G. Consolini, P. D. Ditlevsen, R. V. Donner, and V. Quattrociocchi, "Multiscale measures of phase-space trajectories," Chaos **30**, 123116 (2020).

[15] D. Faranda, V. Lucarini, G. Turchetti, and S. Vaienti, "Generalized extreme value distribution parameters as dynamical indicators of stability," Int. J. Bifurcation Chaos **22**, 1250276 (2012).

[16] M. D. Chekroun, E. Simonnet, and M. Ghil, "Stochastic climate dynamics: Random attractors and time-dependent invariant measures," Physica D **240**, 1685–1700 (2011).

[17] N. E. Huang, Z. Shen, S. R. Long, M. C. Wu, H. H. Shih, Q. Zheng, N. C. Yen, C. C. Tung, and H. H. Liu, "The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis," Proc. R. Soc. London Ser. A **454**, 903–998 (1998).

[18] N. Rehman and D. P. Mandic, "Multivariate empirical mode decomposition," Proc. R. Soc. London Ser. A **466**, 1291–1302 (2010).

[19] T. Alberti, R. V. Donner, and S. Vannitsem, "Multiscale fractal dimension analysis of a reduced order model of coupled ocean-atmosphere dynamics," Earth Syst. Dyn. **12**, 837–855 (2021).

[20] V. Lucarini, D. Faranda, G. Turchetti, and S. Vaienti, "Extreme value theory for singular measures," Chaos **22**, 023135 (2012).

[21] E. Ott, *Chaos in Dynamical Systems*, 2nd ed. (Cambridge University Press, 2002).

[22] P. Grassberger and I. Procaccia, "Measuring the strangeness of strange attractors," Physica D **9**, 189–208 (1983).

[23] L. Schimansky-Geier, A. V. Tolstopjatenko, and W. Ebelin, "Noise induced transitions due to external additive noise," Phys. Lett. A **108**, 329–332 (1985).

[24] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, "Stochastic resonance," Rev. Mod. Phys. **70**, 223–287 (1998).

[25] J. P. Eckmann and D. Ruelle, "Ergodic theory of chaos and strange attractors," Rev. Mod. Phys. **57**, 617–656 (1985).

[26] V. Lucarini, "Stochastic perturbations to dynamical systems: A response theory approach," J. Stat. Phys. **146**, 774–786 (2012).

[27] B. Eckhardt and G. Ott, "Periodic orbit analysis of the Lorenz attractor," Z. Phys. B: Condens. Matter **93**, 259–266 (1994).

[28]C. C. Maiocchi, V. Lucarini, and A. Gritsun, "Decomposing the dynamics of the Lorenz 1963 model using unstable periodic orbits: Averages, transitions, and quasi-invariant sets," Chaos **32**, 033129 (2022).
[29]A. Gritsun and V. Lucarini, "Fluctuations, response, and resonances in a simple atmospheric model," Physica D **349**, 62–76 (2017).
[30]V. Lucarini, "Evidence of dispersion relations for the nonlinear response of the Lorenz 63 system," J. Stat. Phys. **134**, 381–400 (2009).
[31]D. Faranda, G. Messori, and P. Yiou, "Dynamical proxies of North Atlantic predictability and extremes," Sci. Rep. **7**, 41278 (2017).
[32]R. V. Donner, Y. Zou, J. F. Donges, N. Marwan, and J. Kurths, "Recurrence networks—A novel paradigm for nonlinear time series analysis," New J. Phys. **12**, 033025 (2010).
[33]R. V. Donner, M. Small, J. F. Donges, N. Marwan, Y. Zou, R. Xiang, and J. Kurths, "Recurrence-based time series analysis by means of complex network methods," Int. J. Bifurcation Chaos **21**, 1019–1046 (2011).
[34]T. Alberti, F. Daviaud, R. V. Donner, B. Dubrulle, D. Faranda, and V. Lucarini, "Chameleon attractors in turbulent flows," Chaos, Solitons Fractals **168**, 113195 (2023).