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The paths of nine mathematicians to the realm of dynamical systems

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ABSTRACT
This article is devoted to the first steps of nine mathematicians from five countries on their path to mathematics, chaos and discrete dynamical systems, some from early childhood. In these life stories, the names of outstanding mathematicians arise, crisscrossing the nine stories in unexpected ways. These mathematicians also interacted with each other, forming an intriguing social network world-wide, across all borders of nationality and languages.

KEYWORDS
History of dynamical systems, life of mathematicians, Fields medalist, Abel Prize, Belykh attractor, Chua attractor, Hénon map, Lorenz attractor, Lozi map, Shilnikov attractor

1. Introduction by René Lozi

This article is dedicated to the first steps of nine mathematicians from five countries on their path to mathematics, chaos and discrete dynamical systems, some beginning from their early childhood. In these life stories, the names of famous mathematicians

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arise, crisscrossing the nine stories in unexpected ways.

In March 2021, I was appointed associate editor of the International Journal of Difference Equations. Saber Elaydi the editor in chief asked me to edit a special issue dedicated to my area of research. He suggested the title “Lozi, Hénon and other chaotic attractors, theory and applications”. I asked my long-time colleagues and friends Lyudmila Efremova, Mohammed-Salah Abdelouahab, Safwan El Assad and Michal Pluhacek to help me in this task, by agreeing to be co-editor.

For two years, we sent hundreds of emails to potential authors in more than twenty countries around the world to finally publish this special issue of 32 articles. During this period, inspired by the collective article “Some elements for a history of dynamic systems theory” [110] in which around fifteen authors, including me, described their early involvement in research on chaos, I asked several of the oldest authors of the special issue to describe their path to mathematics and discrete dynamical systems without setting a start date for their reminiscence. Some declined the invitation, others accepted after very long discussions. I addition I asked also three recipients of the pretigious Bernd Aulbach prize of the International Society of Difference Equations to contribute: Michał Misiurewicz who in 1979 coined the name “Lozi map”, Jim Cushing and Saber Elaydi (the fourth recipient Laura Gardini was already involved as author in the special issue). All of them were asked to focus on their early times without any other indication. Therefore, this article has to be considered as a companion of the special issue.

A history of dynamical systems and chaos theory was provided by Aubin and Dahan-Dalm’edico, focusing on three important contributors from the 1960s (Smale, Lorenz and Ruelle) [19]. However, their article is based on the publications of the scientists who constructed this domain of mathematics and linked sciences and does not highlight their motivations. From another point of view, the famous essay entitled “On How I Got Started in Dynamical Systems” by Steve Smale [193], describes his encounters with the mathematicians who inspired him and made him want to study chaos and invented the “horseshoe” (between other important discoveries he made). It is with this in mind that this recollection article was designed.

Mathematical research is often perceived as an exclusively male domain, despite the significant contributions made by women to the discipline. This misconception is partly due to their excessive modesty, which prevents them from promoting their discoveries in society, but also to the weight of prejudices that have affected them in society for centuries. This article is no exception to this imbalance, but I am delighted that a third of the contributions are by women.

This article is organized as follows: in section 2, Michał Misiurewicz describes his love for mathematics since his early childhood (starting counting when he was three years old and reading math text book for the first garde at the age of five and solving “problems” from that book). He emphasizes the importance of the International Mathematics Olympiads to him during his high school years (calling them a “great adventure”) and the brilliant idea of creating “special groups” for gifted students at the university level. In 1966, he was admitted as a student to the Faculty of Mathematics and Physics of the University of Warsaw, field of mathematics, resisting to his Grandfather who tried to convince him that he should go to the Technical University instead, and become an engineer. When he was a second year student, Włodzimierz Holsztyński, a topologist, started to mentor him. In particular, giving him a problem in lattice theory. He solved the problem which resulted in his first publication, in 1969. He describes the origin of dynamical systems in Warsaw around fifty years ago due to the influence of Stanisł aw Mazur, Andrey Kolmogorov and Yakov Sinai on Wiesław Szlenk and Karol Krzyżewski. Jan Maria Strelcyn was also member of the Warsaw group before leaving to France. He explains how he switched from topology to dynamical systems (but still working between both specialties), due to the move of Holsztyński to the USA, leaving Poland. and how he remained stuck with one-dimensional dynamics having about two-thirds of his papers about it. Eventually he gives a practical advise for young mathematicians.

Laura Gardini, in Section 3 recalls that her decision to study Mathematics was taken when she was 12 years old. There was not a specific event that led to her decision, she simply loved Mathematics and realized that she was happy studying it, and that it was easy for her. She was living in a very small village, and she knew that her goal was to convince her parents of her will to continue to study in a scientific lyceum (the closest was in Ravenna, 10 km, far from her village) and then to study Mathematics at the closest University which was in Bologna. When she graduated (with honors) in 1975, she found research position at ENI (Ente Nazionale Idrocarburi) group. After reading some books at the beginning of the eighties, by Thom [197], Zeeman [209], Poston and Stewart [151], Collet and Eckmann [45], Guckenheimer and Holmes [79], she immediately loved dynamical systems and therefore decided not to change anymore. Since then it has been her area of research. She resigned from ENI in Summer 1987, studying to win some competition at the University, that happened the year after, and she started as Researcher at the University of Urbino in November 1988. She then came into contact with Christian Mira in Toulouse, and established a friendship and a fruitful collaboration with him and later with Ralph Abraham. She began other long collaboration with Yuri and Vladimir Maistrenko and Iryna Sushko. In Urbino (Italy) she invited many other well famous scholars, in particular, Leonid and Andrey Shilnikov. In Minneapolis in 1995, she collaborated with Christos Frouzakis and Ioannis Kevrekidis. Since then the number of her collaborators working on smooth and non smooth maps has constantly been increasing.

Vladimir Belykh says in Section 4 that from 1961 to 1965, he studied mathematics and physics in the Chair of Oscillation Theory and Automatic Control, named after its founding director, Alexander A. Andronov, at Gorky State University (nowadays Lobachevsky State University of Nizhny Novgorod). His professors were Dmitry A. Gudkov, Yuri I. Neimark, Nickolai A. Fufaev, and Nickolai A. Zheleztsov, the brilliant team of Andronov's disciples. Later he was accepted at the Research Institute of Applied Mathematics & Cybernetics were he met Leonid P. Shilnikov and became friend with him. He met also Alexander Sharkovsky in 1965 who was later external reviewer of his Ph.D. thesis in 1972. During his long career not yet finished, he proved many important results and especially invented the "Belykh attractor" (the name was coined by Valentin Afraimovich) in September 1976, which was remarked by Yakov Sinai. Dmitri Anosov who served as an opponent for his "Doctor of Sciences" degree thesis in 1983, strongly recommended to include the description of this map and its detailed analysis in the thesis. The same year he met Vadim Anishchenko at a conference on the Oka river and both became quickly life-long friends. He collaborated also with Leon Chua and visited him at UC Berkeley and later with many other outstanding researchers like Martin Hasler at École Polytechnique Fédérale de Lausanne (EPFL).

The research of Eckehard Schöll encompasses mathematics and physics. He recalls in Section 5, that as a school boy in a small German town, Nürtingen near Stuttgart in Southwest Germany, he was interested in a very broad spectrum of subjects, ranging from Latin, English, French, German via History, Music, Art to Mathematics and Physics. But when he passed the final school exam (Abitur) with best grades in all subjects in 1970, it was clear that he wanted to study Physics at university. In a Physics Colloquium in 1974 in Tübingen he heard a lecture by Werner Heisenberg and was fascinated by his personality. He attended a mathematically oriented conference "Rencontre entre mathématiciens et physiciens théoriciens" in Strasbourg in 1976 when he was still an undergraduate student and was very impressed by a talk by David Ruelle. In summary, he studied a wide selection of basic and special courses in theoretical and experimental physics and pure and applied mathematics, and moreover also history of arts, musicology, and philosophy. After finishing his Diploma degree in physics in 1976, he started to work with Peter Landsberg in Southampton on combined generation and recombination processes in semiconductor with nonequilibrium thermodynamics and a nonlinear dynamical systems approach. There he met David Chillingworth who came from the Warwick dynamical systems group and put this research into the more mathematical framework of bifurcation theory and catastrophy theory. He completed a Ph.D. Thesis in Applied Mathematics in Southampton on "Nonequilibrium Phase Transitions in Semiconductors" in 1978. After that he moved to Aachen and prepared another Doctoral Thesis (Dr. rer. nat.) in Physics, with Prof. Friedrich Schlögl which as mentor and completed it in 1981. He describes then his outstanding career in details and reveals his secret in the life: since he has spent most time with young students, he has also stayed young!

In Section 6, Galina Strelkova says that during her high school years, her interests were quite broad. She liked mathematics and physics, but was very attracted to medicine, reading on surgery with great interest and sometimes being even allowed to attend operations in clinics (her mother was an anesthesiologist). In addition, over all her school years, she also seriously studied music (piano and vocals), composed music for poetry, took part in various competitions and became their laureate several times. However, by the time she graduated from high school, she definitely decided to become a school teacher in mathematics, physics and computer science. After graduating from high school she went to Saratov (a big regional center at the Volga river, 260 km far from her native place) to enter the Department of Radiophysics of the Faculty of Physics of Saratov State University. Since then her scientific life has been inextricably linked with this department. In the early 90s Professor Vadim Anishchenko director of the Nonlinear Dynamics Laboratory introduced her to dynamical systems and she became a member of this laboratory. After defending her Ph.D. thesis she was invited by Professor Jürgen Kurths in his Nonlinear Dynamics Working Group in Potsdam University and published several joint papers with him. Thanks to the initiated collaboration and the activities of professors E. Schöll and A. Zakharova since 2015 the Nonlinear Dynamics group was involved into a very interesting field of research devoted to the study of complex spatiotemporal structures, such as chimera states in networks of coupled nonlinear oscillators. This collaboration was very fruitful and successful and is still continuing and resulted in numerous joint publications, exchange research visits, plenary and invited talks at different international conferences and workshops, as well as participation in several international scientific grants and projects. After defending her Doctor of Sciences Thesis (Habilitation) on October 1, 2020 and the death of Vadim Anishchenko on November 30, 2020 G. Strelkova headed the Radiophysics and Nonlinear Dynamics Department of Saratov State University and mentored several Ph.D. students.

Section 7, is devoted to the particular history of Saber Elaydi, who from a young age, displayed an insatiable thirst for knowledge that transcended the confines of his circumstances. In a world plagued by uncertainty and limited resources, he sought solace and inspiration within the realm of mathematics. The numbers and theorems became his refuge, offering a sanctuary where possibilities knew no boundaries. He delved into the world of mathematics, voraciously consuming every piece of knowledge he could find. His perseverance caught the attention of mentors and benefactors who recognized his talent. This unwavering dedication earned him a scholarship that paved the way for formal education, propelling him beyond the confines of the refugee camp. In 1978, he received his Ph.D. from the University of Missouri under the guidance of Ping-Fun Lam and David Carlson. His dissertation, titled "preferred sets in topological dynamics", was focused on transformation groups. After his graduation, he was hired as an assistant professor at Kuwait University where he worked with three different researchers H. Farran, F. Dannan and S. Kaul. In 1983, he moved to Case Western Reserve University, Cleveland, Ohio. He worked with Otomar Hajek on the study of dichotomy and trichotomy of nonlinear differential equations, two of the most important asymptotic properties of dynamical systems. However, he was eager to return closer to his original interest. And in 1987, he found it in difference equations and discrete dynamical systems. In 2001, together with Bernd Aulbach they created the International Society of Difference Equations (ISDE). This creation was announced during ICDEA 2001 at the University of Augsburg. Throughout his career, he has held several key positions, including Co-Editor-in-Chief of the Journal of Biological Dynamics (JBD) and Editor-in-Chief of the Journal of Difference Equations and Applications (JDEA). He mentored numerous Ph.D. students and authored various publications in discrete dynamical systems and Ecology including *Discrete Chaos: Applications in Science and Engineering*, *An Introduction to Difference Equations*, *Upside-Down: The Interplay between Life and Chaos*, and co-authored with Jim Cushing *Discrete Mathematical Models in Population Biology: Ecological, Epidemic, and Evolutionary Dynamics*. Two more people have had great influence on his research: Lord Robert May and Jim Yorke.

In Section 8 Jim Michael Cushing remembers that he was so enamored with mathematics when he was student in the late 1950's in Cheyenne, Wyoming, that he would check out the text book from the local Carnegie library during the summer recess before he took a math course in order to teach himself, as best he could, the topic prior to the upcoming school year. He adds that, given the lack of activities available to young people in such a small town, he also spent a great deal of time studying piano performance, practicing sports (basketball, football, and baseball), and reading philosophy (a lot of Aristotle). In College and university of Colorado in Boulder he focused on courses in physics, chemistry, and geology as well as astronomy and psychology. Eventually during his senior year, he enrolled in postgraduate mathematics classes and participated, as part of the University's team, in the preeminent national William Lowell Putnam Mathematical Competition for undergraduate students. Having obtained a National Defense Act Fellowship he became in 1964-68 a graduate student in applied mathematics at College Park, Maryland, where he met Jim Yorke. In this Section he describes in details his outstanding career which followed a long path from his Ph.D. on the Levi-Civita's conjecture concerning deep water waves, with Monre H. Martin as advisor, to theoretical ecology. He explains in particular how was created the Beetle Team consisting of Bob Costantino, Bob Desharnais and him together Brian Dennis and later Shandelle Henson and Aaron King in 1989-90. This team gained the support of NSF for over a decade, during which they developed and parameterized a discrete time (Leslie) model using historical data, studied the predictions of the model using analysis and numerics, determined bifurcation sequences that included chaotic dynamics and experimentally feasible manipulations that place replicated cultures along that route-to-chaos, and statistically validated (without re-parameterization) the model predictions against the observation data obtained. He adds that during his travels on this long career path, he never stopped playing the piano.

In Section 9, Lyudmila Efremova native of Nizhny Novgoro (previously Gorky between 1932 to 1990 describes) first highlights the well-known Andronov's school of nonlinear oscillations and dynamical systems. She recalls that the Andronov's great merit is the use and development of fruitful, but forgotten (by 20s – 30s of 20th century) Poincaré's ideas in the theory of differential equations for the needs of radiophysics and the theory of radio transmitters. In 1956 the Gorky University was named after the great Russian mathematician, a native of Nizhny Novgorod, N.I. Lobachevskii. In the struggle between her two serious hobbies, mathematics and music, mathematics won, and she entered the Faculty of Mechanics and Mathematics of the University. When she was a 2nd year student, she had a huge impression reading the Sharkovsky's paper [187] which was submitted to her by Dr. Rakhmankulov who give her next year the subject of her Ph.D. Now after many years she can say that her first impression of Sharkovsky's Theorem determined her scientific biography. She says that her student years passed with a continuous feeling of happiness of learning new things, listening lectures given by major mathematicians and physicists, as well as famous musicologists at the Gorky Conservatory on music theory. During the second year of her Ph.D., she was invited by Sharkovsky to present her results at his seminar at the Institute of Mathematics of the Academy of Sciences of Ukraine and recommended her work to the Programme Committee of the IX International Conference on Nonlinear Oscillations (ICNO - IX, Kiev, 1981). It was the first major conference in her life and Professor Plykin was the chairman of the section where she presented her report. She later saw the poems written by him and was astonished by the fact that a mathematician could be a poet. Professor Otrokov and Dr. Rakhmankulov, were her scientific supervisors, and Sharkovsky and Belykh were the referees of her Ph.D. After that, another period was coming, in which the most interesting areas of activity seemed to her, firstly, the applications of one-dimensional dynamics to the study of discrete dynamical systems on manifolds of dimension at least two; and, secondly, the creation of dynamical systems theory on complicated one-dimensional ramified continua, which do not allow order topology. She worked on these topics under the guidance of Professors Anosov and Stepin. Eventually, Sinai chaired the jury of her Doctor of sciences thesis.

In section 10, René Lozi indicates that like Misiurewicz, he was counting many things around him in his early childhood. He started his studies at the University of Nice in 1967. He was fascinated by particle physics. After being graduated in fields and particles, he changed his mind and became numerical analyst. The subject of his Ph.D. was related to bifurcation theory, which was near unknown in France in 1972, among mathematicians. In 1974, a group of researchers in bifurcation was constituted in the mathematics department in Nice, around Gérard Iooss. With this group he attended a Conference in Roma where he heard David Ruelle speak about the Hénon map. He soon proposed a piecewise linear version of this map, known now as "Lozi map" in 1978. After that, his interest shifted to research of realistic model of chaos, first studying slow-fast differential equations after his meeting with Claude Lobry and introducing the theory of confinors in place of attractors. René Thom was interested by his morphological modeling of the Lorenz equation and was in the jury of his doctoral thesis, along with Michel Hénon. He met in 1986 Leon Chua and studied the "Chua circuit" for many years with Shigehiro Ushiki from Kyoto university. He worked also with many renowned scholars like Alexander Sharkovsky, Guangrong Chen, Leon Chua.

2. My road to dynamical systems by Michał Misiurewicz

In order to arrive to Dynamical Systems, of course I had first to arrive to Mathematics.

My first encounter with mathematics which I remember was counting the cars of a moving train. I do not remember how old I was then. At this moment, let me make two digressions.

The first one is about counting. The urge to count various things remained with me for my whole life. In particular, even now when I see a moving train I start counting cars.

The second digression is about my memory. It seems that it works in a not quite typical way. On one hand, people often are surprised that I remember so many songs and so many jokes. On the other hand, I remember only very small fragments of all years of my life. And problems with remembering things appear also in mathematics. For instance, I do not remember the direction of the inequality between the arithmetic and geometric means. I have to check it for 1 and 4, and only then I recover the direction. Similarly for the Fatou's Lemma [65]; I have to go through an example to know which way the inequality goes. And from time to time funny things happen. Once I was listening to a conference talk and noticed a very nice trick. After the talk I asked the speaker who is the author of this trick. And he answers "I do not know, but I learned it from you."

Fortunately, my Mother was keeping a dairy, so I know how I was progressing. I started counting when I was three years old. By the age of five, when I learned to read, my favorite book was the math textbook for the first grade; I was solving the "problems" from that book. Does this mean that it was then when I already become a mathematician?

However, during vacations I forgot almost all that I learned before. But later I either recalled quickly what I knew, or I learned it again.

At a certain moment I started to read the book *Lîlâvatî* by Szczepan Jeleński[1] and to solve problems from that book. I was fascinated by the book. In particular, I became interested in the story where somebody asks as a prize 1 grain on the first square of a chessboard, 2 on the second one, 4 on the third one, etc. (exponentially). I took a notebook and started to compute the consecutive powers of 2. The notebook survived (although I do not know where it is now), so much later I checked my computations with a calculator. It turned out that somewhere around 2^{30} I made an error, and of course all further powers was also incorrect.

After reading about the Fermat's Great Theorem, of course I decided to think about it. In such a way I discovered by myself the formula $a^2 - b^2 = (a-b)(a+b)$. So it seems that I was already a mathematician by that time.

In the high school I had a partner for mathematics. Péter Szeredi, a son of Hungarian diplomats, living at that time in Warsaw, was in the same class. He was subscribing the Hungarian journal *Matematikai Lapok*, with lot of interesting problems.

Finally, a great adventure – olympiad. When I was in the ninth grade (high school at that time consisted of grades 8–11) I took part in the Mathematical Olympiad. It consisted of three stages. I managed to qualify to the third stage. This already gave me the right to become a student of mathematics at any Polish university without the entrance exam; however, I had to graduate from the high school first. At the third stage competition I did not do well, in particular because I did not know trigonometry yet. I did not even know what sine and cosine were.

Next year (1964-1965), next olympiad, tenth grade. This time I became one of the laureates. What is more important, I qualified to the Polish team for the International Mathematical Olympiad (IMO). There was a "training camp" for the team. I still remember the number of the bus I had to take each day to attend it.

IMO took place in Berlin. I got a third degree diploma – now it is called a bronze medal, but at that time there were no medals. I got 29 points out of 40 possible. I still have an interesting photo, made on a boat (see Figure 1). A group of participants, that include myself and Pavel Bleher. I could not guess at that time that in a distant future I will be working with him at the same Math Department in Indianapolis for 27 years. Another memory – a joke problem: evaluate the determinant of a big complicated matrix. The matrix was so big that it was not easily visible that it was a 13×14 one.

One more olympiad, eleventh grade. Again I was a laureate, but this time without a diploma, because one can get only one diploma at the Polish Mathematical Olympiad. Years later I could see the grades I got. Each problem was graded by three people, and then the final grade was negotiated. From one problem I got (on the scale from 2 to 5) 2, 2+, and 5 with an exclamation mark (an extra praise). The final grade was not given, because nothing depended on it. I still do not know whether my solution was correct.

In order to graduate I have to pass school-leaving exams. The math exam was of course very easy for me, so the only thing I remember is that a sparrow flew into the

[1] See `https://polona.pl/item/lilavati-rozrywki-matematyczne,OTgwNzY0NjQ`; it seems that it has not been translated to English; a pity.

Figure 1. A photo from the 1965 International Mathematical Olympiad. Among people standing I am the rightmost one and Pavel Bleher is the leftmost one.

room through an open window and left a “present” on my table.

Again I was in the Polish team for the International Mathematical Olympiad (this was 1966). Again the training camp. I remember two things from it. One is that we tried to construct a theory of half-sets. The basic examples of half-sets were the intervals $(a, a]$ and $[a, a)$.

Surprisingly, much later this turned out to be not a pure joke; I used this idea in several papers (for instance [136] and [138]). The second thing I remember was a problem posed by Włodzimierz Kuperberg about a broken line and n points on the plane. The problem is relatively simple for $n = 1, 2, 3, 4$, but for larger ns it is still open.[2]

IMO that time was in Sofia. The problems were relatively simple, so I managed to get the maximal score, 40 points, and a diploma of the first degree (nowadays a gold medal). Not all of us were that lucky. One of my colleagues immediately after leaving the room (so he already could not change anything) realized that every stick has two ends. When counting the endpoints of segments in one of the problems he forgot about it.

In 1966, using one of the four documents (three for qualifying to the third stage

[2]Define a broken line as a union of finitely many closed straight line segments, and consider only those broken line that are not homeomorphic to a circle. Does there exist a broken line B and n points in the plane such that every straight line through any of those points intersects B at exactly two points? Continuing traditions of the Polish school of mathematics, Włodek set a prize. For a solution for $n = 5$, a half of a chocolate bar. For $n = 6$, a quarter, etc. Then for a solution for every n the prize would be a whole bar.

of the olympiad and one diploma) I was admitted as a student to the Faculty of Mathematics and Physics of the University of Warsaw, field of mathematics. My Grandfather tried to convince me that I should go to the Technical University instead, and become an engineer. Engineers are always needed. In 1917 he lived in Odessa, and after the revolution the bolsheviks not only spared him, but even were paying him in golden rubles for his work as an engineer. I recalled this many years later in Indianapolis, when I was working as a professor of mathematics with tenure, and a nearby factory producing auto parts was closed, flooding the job market with 600 unemployed engineers.

Switching from the school to the university was a great relief. At last I did not have to learn things that I absolutely did not care about. And at the university – mathematics, mathematics, and mathematics. Several additional, unimportant and not taking much time things, but basically only mathematics. And I did not have to be in the school at 8 in the morning.

An idea that some people consider is controversial, but from my perspective was absolutely great, was the creation of so called "special groups." The lectures were held for all students, but for recitations smaller groups were created. While during the lectures only the professor was talking, the recitations were devoted to solving problems by the students. After several first weeks, the students that wanted (and the professors agreed that they were able) to solve more difficult problems were moved to those special groups (see Figure 2). Usually it was not only about more difficult problems, but also more theory.

The people that are against this idea claim that if the best students are kept in the same groups as the rest of the students, they will help those students. This may be partially true, but in such a way the potential of the best students will be wasted. One has to remember that the progress in any discipline is mainly achieved by the best minds in that discipline. And in my 47 years of teaching I saw many times how difficult it is to teach a group of students whose knowledge and abilities are all over the spectrum.

Recitations from Analysis I were conducted by Witold Kołodziej. He was the best teacher among those I meet during my studies. He was demanding; sometimes my work was graded for 3 (in the scale from 2 to 5), but he really taught me Real Analysis (and at the end of the year I got a 5). The material I learned at the first year of studies was approximately the same that I much later taught in the US at the 400 level course in Real Analysis.

A quick comment about the differences between Poland and the US, when it comes to the teaching of the future mathematicians. In Poland one is considered a fully educated mathematician after getting the MS degree; in the US it is the Ph.D. Thus, compared to Poland, the education of a young person in the US is shifted by about 3 years. And so is the choice of their career.

After each semester we had to pass exams. I was usually preparing for the exams with my two colleagues: Michał Krych and Piotr Minc. To prepare for the theoretical part of an exam after a one-semester class we needed one week of hard work. For a two-semester class we needed two weeks. The exams were both written (solving problems) and oral (questions about the theory). However, if one got a 5 both from

Figure 2. The special group in Analysis I, 1966/67. I am sitting on the second row, third from left (in a sweater).

the recitations and from the written exam, usually the necessity of passing the oral exam was waived.

The next digression. A very important thing for a student (and also for anybody in many circumstances) is to make others to have a good opinion on yourself. This helps a lot. For example, one of my younger colleagues in Warsaw took the written part of the exam and in order to not to waste time immediately went for vacations. However, his exam got lost. What to do? Since the professor had good opinion on him, he gave him a 5.

During summer vacations two times (after the first and second years) I attended the Summer Schools in Algebraic Topology in Sopot (at the north of Poland). I remember a lecture by Peter Hilton, where he was talking (in English, of course) about von Neumann algebras and C^* algebras. I was wondering: I knew who von Neumann was, but who was this guy Sistar? The subject of the school was not only algebraic topology. Karol Krzyżewski and Jan Maria Strelcyn lectured on dynamical systems. That was my first encounter with this theory.

When I was a second year student, Włodzimierz Holsztyński, a topologist, started to mentor me. In particular, he gave me a problem in lattice theory (it was really about the classes of open sets and of closed sets in a general topological space). I solved the problem and it resulted in my first publication, in 1969. I remember a funny incident connected with this mentoring. Holsztyński was one of the professors conducting the oral exam in topology. I enter the office, get several simple questions, and receive a 5. But then we start to talk about those lattices. When I am leaving the office, my colleagues waiting for the exam are already trembling with fright. What kind of an exam is it? Misiurewicz knows all the course material, but he is staying there already a full hour!

Holsztyński left Poland for America in 1969. If he didn't, I would probably be a topologist. However, even now I like the topological aspects of dynamical systems.

When I started my third year of the undergraduate studies, Wiesław Szlenk organized a seminar in mathematical analysis. In spite of the name, it was really a seminar in dynamical systems.

The origin of this discipline in Warsaw was quite interesting. At some moment two young Warsaw mathematicians, Wiesław Szlenk and Karol Krzyżewski, were sent by Stanisław Mazur to Moscow for a postdoctoral training. They were specialists in functional analysis and went to Andrey Kolmogorov for the advise of what they should do. Kolmogorov told them that the great days of functional analysis were already over, but there was a new interesting discipline, called dynamical systems (together with the ergodic theory). Then he sent them to Yakov Sinai (who had been his student) to learn it.

A small digression: according to the *Mathematical Genealogy Project*, Kolmogorov had 82 students, and 3901 descendants. Sinai has 39 students and 356 descendants (among them, two grandgrandgrandgrandsons).

Krzyżewski and Szlenk, after learning the basics of the dynamical systems, returned to Warsaw. They were followed by Sinai, who came and gave a cycle of lectures in ergodic theory. The next member of the Warsaw group was Jan Maria Strelcyn, who was taking notes from Sinai's lectures, that were later available as a booklet. Around 1969 Strelcyn left Poland for France, but Szlenk, and later also Krzyżewski, started to build a group in this new area of mathematics. Krzyżewski concentrated on scientific supervision over new students, while Szlenk, with his extraordinary organizational skills, first organized a seminar (which, after many changes, still exists), and then was taking care of the development of the careers of the students. Having many contacts in the East and West, he was sending us to various conferences and trainings. First two students from the new generation were Michał Krych and myself. Next students that joined were Feliks Przytycki and Maciej Wojtkowski, and after them many others.

My next three papers, published in 1970, 1971, and 1973, were already on dynamical systems. According to the *Google Scholar*, they have together 198 citations, and are still being cited. So that was definitely time when I could say that I arrived to Dynamical Systems.

And here I am after the next 50 years.

I started by investigating topological entropy. In 1972, at a so called international course *Global Analysis and Its Applications* in Trieste, I met Karl Sigmund, who introduced me to the results of Rufus Bowen on entropy-expansive maps. I started to generalize this notion and in 1974 wrote a Ph.D. thesis on this generalization. I met Bowen only once, but we corresponded a little, and I still have his letters written by hand (explanation for young readers: this is what you did before the Internet era).

This stayed with me. I wrote 44 papers and one book with the word "entropy" in the title.

In 1976 Michel Hénon published a paper about his attractor [85]. After spending about half a year on trying to prove that it is really an attractor, I decided that temporarily I will investigate interval maps and only when I gain substantial knowledge of them, I will return to the Hénon's attractor. As everybody knows, temporary solutions usually last very long, and so I remained stuck with one-dimensional dynamics. In fact, about 2/3 of my papers are about it. Return to the Hénon's attractor did not happen, although I managed to prove existence of an attractor for similar Lozi maps [135].

However, my start in the theory of interval maps was quite successful. In 1977 I wrote with Szlenk a paper, where we proved some basic results [140].

Here a practical advise for young mathematicians. If you want, without studying a lot of technical tools, to write a paper that will be often cited, you should choose a subject which will soon become fashionable, but at a moment the basic theorems are not proved (or even basic definitions are missing). So you introduce definitions, prove basic theorems, and then sit down and observe how other people continue the research that you started. And from time to time you add some brick to the building that is being constructed.

And the only small problem is how to predict what will become fashionable. In fact, you cannot predict it. But you can try. Most of times you will fail. But maybe sometimes you will succeed?

3. My road to dynamics by Laura Gardini

My decision to study Mathematics was taken when I was 12 years old. There was not a specific event that lead to my decision, I just loved Mathematics, I realized that I was happy when studying it, and it was easy for me. But I was living in a very small village, and I knew that my goal was to convince my parents of my will to continue to study in a scientific lyceum (the closest was in Ravenna, 10 km, far from our village) and then to study Mathematics at the University (the closest University was in Bologna). For them it was a kind of "investment" in my future. At the lyceum I was always the first in Mathematics, and at the University in Bologna I was always among the best students in my class.

At those times, in Italy Ph.D. courses did not exist (they were introduced only a few years later, in 1983) and when I graduated (*cum laude*[3], in 1975) my supervisor told me that he could not have other collaborators, and he referred me to the ENI (Ente Nazionale Idrocarburi) group, which in those years had also a Department devoted to research work. I was employed immediately. However, the research subjects were decided by the Director, and these changed quite often in the years.

At the beginning of the eighties I was asked to study a few recent books by Thom [197], Zeeman [209], Postom and Stewart [151], Collet and Eckmann [45], Guckenheimer and Holmes [79]. In the meantime I also had three years of teaching experience at the Faculty of Engineering in Ancona (teaching rational mechanics in the academic years 1984/85/86/87 since as an ENI researcher, I was allowed to teach with a contract).

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[3] with honors

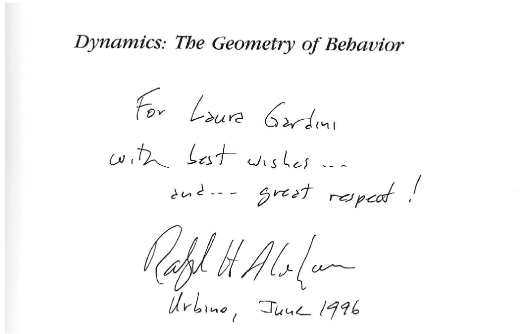


Figure 3. Dedication of Ralph Abraham

That was really a fascinating subject, I immediately loved dynamical systems so I decided not to change anymore. Since then it is my research field. However, to do so, I resigned from ENI in Summer 1987, studying to win some competition at the University, that happened the year after, and I started as researcher at the University of Urbino in November 1988.

As I started working for an Economics Department, my colleagues were proposing me to analyze several models, all systems in discrete time, that is, maps, and mainly noninvertible maps in high dimensions. It was a subject not so widespread in the literature of those years. Fortunately I have seen the books by Gumowski and Mira [80] and [81], in particular the one in French, and Mira [133], and several articles written by them, that pointed me in the right way. When I was preparing the paper [74], at the beginning of 1991, I wrote a letter to Professor Mira, since I thought that there were new dynamic behaviours, related to the saddles and their homoclinic bifurcations, that could be of interest also to him. He was very kind, inviting me to Toulouse, and I visited him in the spring time. That started my friendship and collaboration with Christian Mira.

He also invited me to participate in the *European Conference on Iteration Theory* held in Lisbon in September 1991, organized by Sousa Ramos. There, for the first time I met many researchers, besides Sousa Ramos, who were quite famous in Dynamical Systems, including scholars such as Alexander Sharkovsky, Jaroslav Smital, Sergei Kolyada, Ludomir Snoha, Francisco Balibrea, Jaume Llibre.

There I presented a work showing how homoclinic bifurcations that are not possible in invertible maps may occur in noninvertible ones involving the critical curves[4].

In other words, the first homoclinic bifurcation of a fixed point or of a cycle which is a repelling focus or a repelling node occurs via critical points, i.e. , via a contact of the absorbing area delimited by the critical segments and the stable set of the cycle (which in case of an expanding cycle consists in its pre–images). In the same way, the first homoclinic orbit of a saddle cycle in a region of the phase plane with several preimages can occur via contact of the critical segments with the stable local saddle manifold [75].

This was my favourite subject for a couple of years. In 1992, I presented further re-

[4]Recall that the existence of a homoclinic orbit, in invertible and noninvertible maps, is a proof of the existence of invariant chaotic sets of the map.

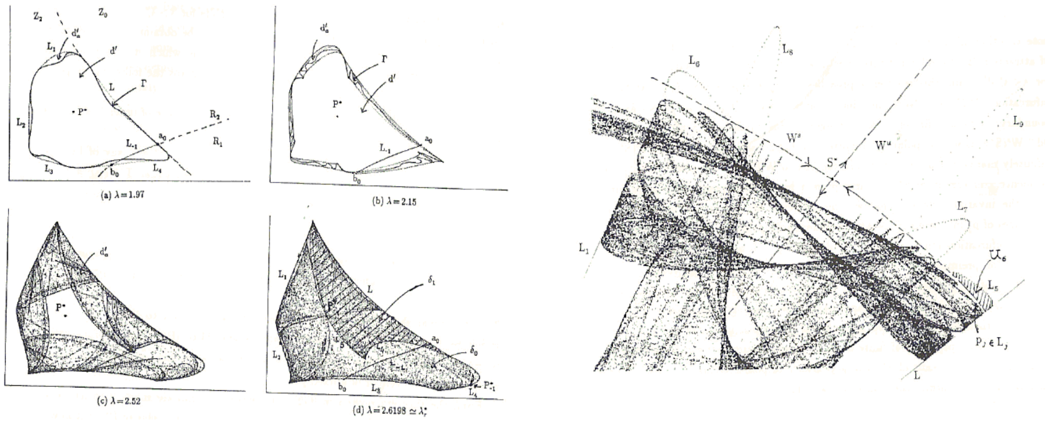


Figure 4. Critical curves and homoclinic tangency with the stable set of a saddle fixed point.

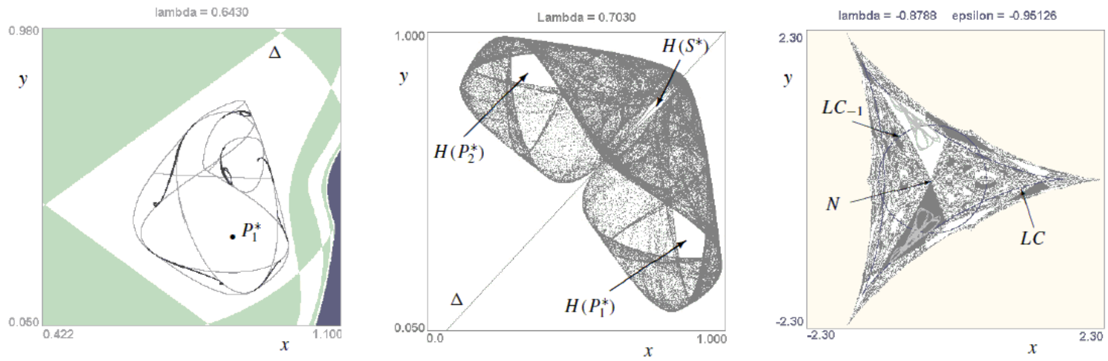


Figure 5. Examples of absorbing areas, snap-back repeller bifurcations and contact bifurcations.

sults, at two other conferences, one in Germany and one in Italy where, in addition to Christian Mira, Ralph Abraham was also invited. Abraham, with whom I also began a long friendship and collaboration, encouraged my research, and was fascinated by the dynamics of non–invertible maps. He was visiting professor in Urbino for several periods (six times from 1992 to 2000). I loved his book [5], which he gave me (with dedication, see scanned page Figure 3). We published together a textbook for beginners [4].

In the same year 1992, Christian Mira invited me to coauthor a book on noninvertible maps. There were so many open problems in two-dimensional noninvertible maps that we were fascinated and curious to understand the bifurcation mechanisms leading to different effects. Disconnected basins of attraction, loops on the critical curves, loops on a closed invariant curve after a Neimark-Sacker bifurcation (destroying its smoothness), contact bifurcations involving the critical curves, leading to chaotic repellors, or expansion and merging of attracting sets, and not only in smooth maps, but also in nonsmooth ones (see Figures 4, 5 from papers of the year 1994).

All this, for the book, required a lot of work, and Mira and I met (in Toulouse or Urbino) several times, at least twice a year, up to its publication, in 1996 [134] (at that time we had exchanges only via fax and letters, we started with e-mails only in the fall of 1996).

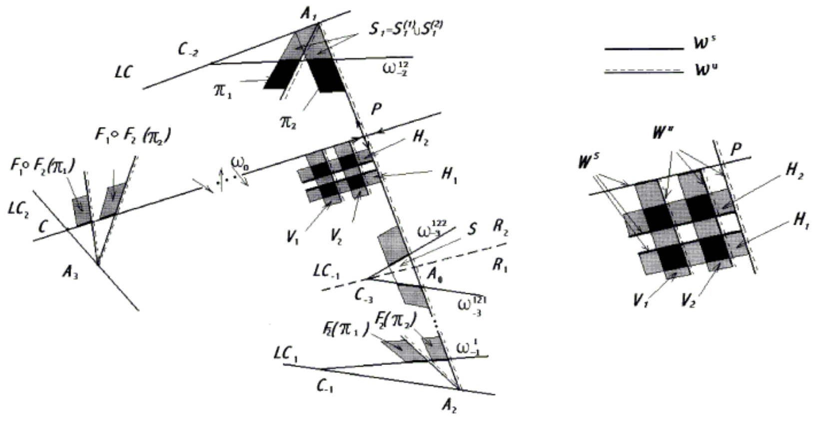


Figure 6. Details of the mechanism of the horseshoe in a noninvertible map.

The homoclinic bifurcations of expanding points and saddles were of interest also for other researchers, in particular Yuri and Vladimir Maistrenko, and the young Iryna Sushko (Ph.D. student of professors Yuri Maistrenko and Alexander Sharkovsky, in Kyiv), with whom I started a long collaboration. They were experts on one-dimensional smooth and piecewise smooth maps, and were starting to investigate the dynamics of a two-dimensional noninvertible piecewise linear map. Iryna was preparing her Ph.D. thesis on that map, and they visited Urbino for some weeks every year, from 1993 to 1997. There is a paper that we liked a lot, [128], in which it is also shown how the Smale-horseshoe mechanism works for the first homoclinic bifurcation of a saddle fixed point in a noninvertible map, involving the critical lines (see Figure 6).

In May 1994 Maistrenko and Sushko organized a conference “Differential Equations, Bifurcations and Chaos”, at Katsiveli, in Crimea, and there for the first time I met many other well known scholars, in particular, Leonid and Andrey Shilnikov. The flight to Katsively from Kyiv deserves a movie. Although it was in May when we left Kyiv it was snowing, and the plane was an old model from the Second World War (I jokingly called it “the albatross”). Inside it was windy and cold, and to relax we had some vodka to drink, so at the end we were all happy. We have to recall that those years were soon after the Ukraine became independent, and we were conscious that everything was difficult. In particular, we had our social dinner without electricity, but the candles worked perfectly. The buildings sometimes had no heating, but the academic conferences were very impressive, and that was enough. We became friends also with Leonid and Andrey Shilnikov. At the time, Leonid Shilnikov was studying a three-dimensional map which, in a degenerate case, was reduced to the two-dimensional quadratic map that we and Mira have used in our book for many examples. So I prepared a kind of working paper collecting relevant bifurcations occurring in that map, and I showed it to Leonid at the workshop *Noninvertible Dynamical Systems* organized by Christos Frouzakis and Ioannis Kevrekidis in Minneapolis (in March 1995, see Figure 7).

Also at that conference (my first visit to the United States) I met many famous scholars. With Frouzakis and Kevrekidis we started to work on a common paper (see [73] and Figure 8 from it) since they were also studying a discrete-time system in which noninvertibility led to particular dynamic behaviours.

Soon after, in 1995, Leonid and Andrey Shilnikov visited me in Urbino (see Figure

Figure 7. Workshop "Noninvertible Dynamical Systems", Minneapolis, March 14-18, 1995. First line from botton, from the left, Andrey Shilnikov, Yuri Maistrenko, my son Martino, and Christian Mira. Second line from the bottom, from the left, Leonid Shilnikov, sitten behind him is Daniele Fournier Prunaret, I am behind Mira, close to my husband (with glasses) and behind him Ralph Abraham.

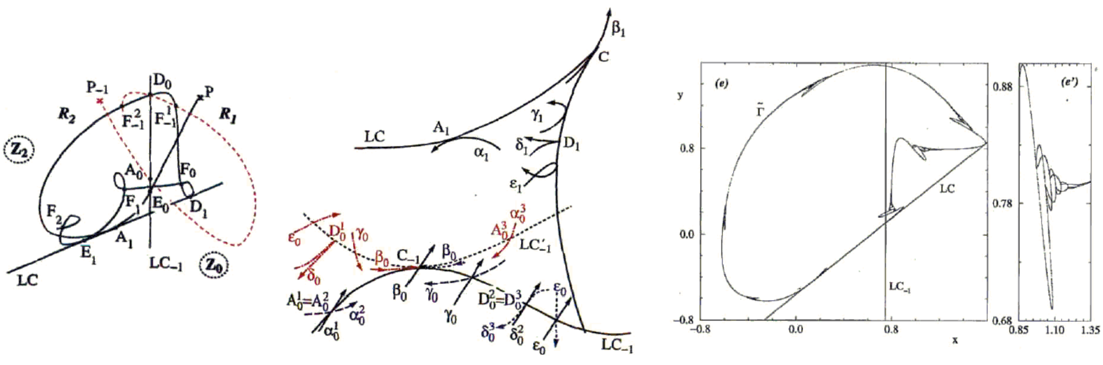


Figure 8. Examples of loops occurring on a closed invariant set of a noninvertible two-dimensional map.

Figure 9. Urbino, June 1995, with Andrey and Leonid Shilnikov.

Figure 10. Urbino, September 96, ECIT conference. From left to right, Leonid Shilnikov, Yuri Maistreno, me, Iryna Sushko, Vladimir Maistrenko, Alexander Kopansky.

9), and I remember that I was so proud and happy when Leonid asked my to show the proof of my paper [76] on the homoclinic bifurcations of expanding cycles in $\mathbb{R}^n$, as he had heard about this work at another conference, mentioned by Ricardo Viana, and he was interested in those homoclinic bifurcations and orbits.

Moreover, the now famous *Blue sky catastrophe model* was presented (for the first time outside Nizhny Novgorod) by Leonid Shilnikov at the *European Conference on Iteration Theory* that I organized in Urbino in September 1996 (see Figure 10). At the same time, I worked closely with Christian Mira, not only on the book, but also on numerous articles, until his retirement (in 2000). Since then, the number of my collaborators working on smooth and non-smooth maps has grown steadily.

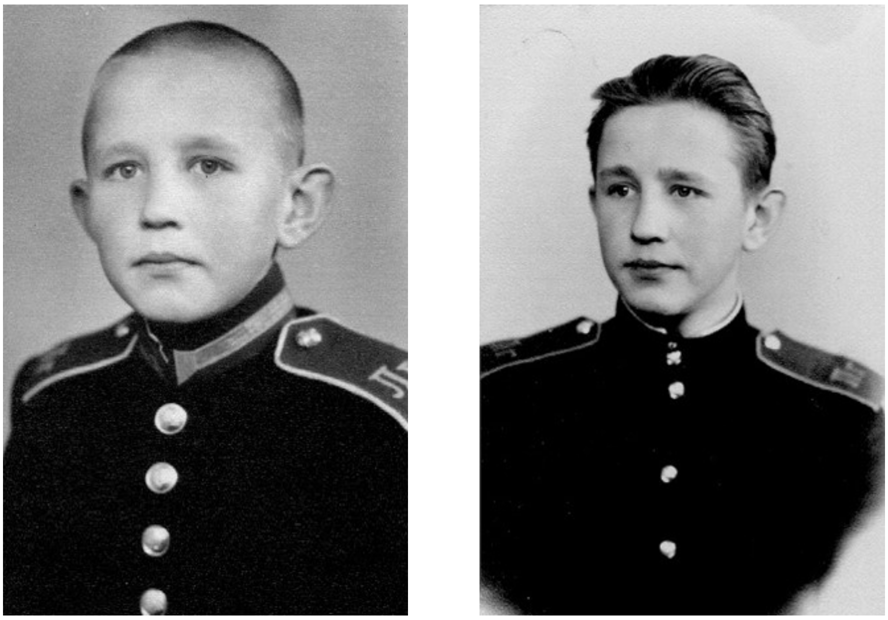
4. My journey into the Chaos Theory: Homoclinic bifurcations and Belykh attractor by Vladimir Belykh

Inspired by the famous essay titled "On How I Got Started in Dynamical Systems" by Steve Smale [193], I would like to share my story on how I embarked on my journey into the Chaos theory. However, in order to arrive to Chaos, of course I had first to arrive to Mathematics. Like Misiurewicz, I have to say few words about my first school years.

After completing elementary school in the Gorky region, at 12, I was accepted into an elite boarding military school in Saint Petersburg. This was a one–of–a–kind college preparatory Suvorov's school for future military, foreign affairs, and intelligence careers. I studied there for five years between 1955 and 1960 and was destined to continue my path in the University of Saint Petersburg military branch (see Figure 11).

In 1960, Nikita Khrushchev (first secretary of the Communist Party of the Soviet Union from 1953 to 1964), initiated massive reorganizations of the Soviet Army and started significant cuts that led to the boarding military school's closure. As a high–school junior, I returned to Gorky. I graduated from high school in 1961 and was accepted into Gorky State University (nowadays Lobachevsky State University of Nizhny Novgorod).

I can say that my academic voyage started there. From 1961 to 1965, I studied mathematics and physics in the Chair of Oscillation Theory and Automatic Control, named after its founding director, Alexander A. Andronov. I was fortunate to take classes from distinguished professors like Dmitry A. Gudkov, Yuri I. Neimark, Nickolai A. Fufaev, and Nickolai A. Zheleztsov, the brilliant team of Andronov's disciples. In the meantime, in 1964, the University established a new large research unit, the Research Institute of Applied Mathematics & Cybernetics (NII PMK as abbreviated in Russian), directed by my future Ph.D. adviser, Liudmila N. Belyustina. As a master's student, I informally became a part of this institute and started working on my diploma under the supervision of Vladimir D. Shalfeev (who was a Ph.D.-level researcher only two years older than me). My diploma research (an analog of an M.S. degree) was on the analysis of homoclinic orbits bifurcations in a two-dimensional pendulum-type dynamical system on a cylinder, modeling a phase-locked loop (PLL), a control system with a nonlinear filter. After graduating in 1966, I joined the NII PMK as a junior scientist. It was usual practice for fresh graduates to be placed in a research institute before they could begin working on their Ph.D. thesis. It was a stroke of luck and a turning point in my academic and personal life that I was accepted into



(a) (b)

Figure 11. (a) Vladimir Belykh at the military school, circa 1955; (b) circa 1960.

the Division of Differential Equations headed by Evgeniya A. Andronova-Leontovich, Andronov's spouse and close collaborator and a sister of another celebrated Soviet physicist, Mikhail Leontovich of Moscow State University. At that time, I continued my work on periodic dynamics of two-dimensional planar systems [22] and had a poor understanding of the emerging concept of "chaos," instead viewed and called "irregular oscillations." Remember that only a few years have passed since the seminal paper's publication on a deterministic nonperiodic flow by Edward Lorenz [113].

I was fortunate to meet Leonid P. Shilnikov (L.P., as friends called him), who worked in the same division. Over time, we became friends, and one day L.P. explained to me his famous theorem on the bifurcation of a homoclinic orbit of a saddle focus, which became a widely-used criterion for the existence of chaotic dynamics in saddle-focus dynamical systems [190]. I felt a lack of knowledge to understand the details of this complicated bifurcation structure fully. Similarly, the depth of my knowledge needed a boost, akin to my earlier encounter with Alexander N. Sharkovsky in 1965 when he presented the proof of his famous Sharkovsky's ordering at a research seminar run by Yuri I. Neimark in NII PMK. I got the main gist of the Sharkovsky's formal result but could not fully understand and appreciate its fundamental implications for observable chaotic dynamics due to my knowledge gap. These experiences were the primary motivations for my Ph.D. thesis, initiated in 1968 under the guidance of Liudmila N. Belyustina at NII PMK's Division of Dynamical Systems and Control Theory, headed by Neimark. My Ph.D. thesis focused on the existence and bifurcations of persistent homoclinic orbits, another criterion for chaos, in periodically driven systems of generalized pendulum equations, all without relying on small parameters or computer assistance. I was not sure how to reach the goal, but after lots of back and forth and head-scratching, I developed a novel method involving auxiliary two-dimensional systems to analyze homoclinic orbits. I first presented my findings at the 1971 conference on ordinary differential equations in Sverdlovsk (nowadays Ekaterinburg) and subsequently published the results [35–37]. A.N. Sharkovsky participated in this conference and attended my talk. He then agreed to be an external reviewer (called an "opponent" in the Soviet system) for my Ph.D. thesis, which I defended in 1972.

1974 marked a significant year as I contributed to the theory of digital PLLs, yielding publications [30, 31] and, along with colleagues Vladimir D. Shalfeev and Valery P. Ponomarenko, the prestigious Lenin Komsomol Prize (a Soviet science award for young researchers under 33, see Figure 12)).

Figure 12. Belyustina's lab picture at the NII PMK circa 1978. Front row (from left to right): Vladimir Belykh wearing the Lenin Komsomol Prize medal, Liudmila Belyustina, Valery Ponomarenko, and Vladimir Shalveev.

In 1975, with my Ph.D. student Vladimir I. Nekorkin, we rigorously proved the existence of Shilnikov's chaos in a three-dimensional autonomous phase system [32].

As part of the European-Soviet scientific exchange program, I spent the 1975-1976 academic year in the Electronics Laboratory at the Danish Technical University (DTU) in Lyngby, a quiet suburb of Copenhagen. During my stay at DTU, I primarily worked with George Bruun and Orla Christensen, two professors of electronics, studying discrete-time digital PLLs. In the meantime, two DTU experimental physicists, Niels F. Pedersen and Ole H. Soerensen, found out that I was an expert in the dynamics of pendulum equations and turned to me with a request to explain a random behaviour of the Josephson junctions' current density-voltage $J - V$ curves they obtained experimentally. Looking at the Josephson junction model equations, I quickly realized that they belong to the same class of pendulum questions studied in my Ph.D. thesis, and all my machinery can be applied to Niels's and Ole's experimental setup. Of the same age and mindset, we quickly became good friends with Niels and Ole and started working on the new project. In the end, we discovered that the mysterious phenomenon of random switching in $J - V$ curves is caused by an infinite, irregular set of bifurcations of homoclinic orbits. We published these results in two companion papers in Physical Review B in 1977 [33, 34], with part I [33] and part II [34] devoted to the autonomous and non-autonomous cases, respectively. These papers were very well-received and cited within both experimental and theoretical physics communities. To the best of our knowledge, our results were the first theoretically validated observations of dynamical chaos in Josephson junctions.

In September 1976, following my return from Denmark, Valentin (Valya) Afraimovich, a Shilnikov's pupil, and L.P. showed me a paper on the original and structure of the Lorenz attractor, which they, together with Vadim Bykov, submitted to a reputable Russian journal [7]. Valya explained to me the action of the discontinuous Poincaré return map they constructed in a qualitative, implicit form. I soon realized that my model of a discrete-time PLL [23] in the case of a piecewise linear nonlinearity becomes a discontinuous 2D map that acts like theirs. I showed this map with a chaotic attractor to Valya and L.P., and Valya said: "Here is the Belykh attractor." L.P. confirmed. It was during this exchange that the Belykh attractor was coined. While there are several variations of the Belykh map [23, 25, 27], its standard form reads [28, 29]

$$\begin{cases} \bar{x} = \lambda x \\ \bar{y} = \gamma y \end{cases} \quad \text{if } L(x, y) \leq 0, \tag{1}$$

$$\begin{cases} \bar{x} = \lambda(x-1)+1 \\ \bar{y} = \gamma(y-1)+1 \end{cases} \quad \text{if } L(x, y) > 0, \tag{2}$$

where $L(x, y) = k(2x-1)+2y-1$. The map is defined on the square $S = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$. The square S is cut into two parts, S_1 and S_2, by the line $L(x, y) = k(2x-1)+2y-1 = 0$ (see Figure 13). The dynamics of the map f is governed by (1) on the lower part S_1, where $L(x, y) \leq 0$, and by (2) on the upper part S_2. Please see the Scholarpedia paper [29] for further details on the map dynamics, and our paper [28] in this special issue for the map derivation from a continuous-time PLL system.

I initially dismissed the importance of this discontinuous, chaotic map due to its apparent simplicity, as its absorbing region contains an attractor composed of only saddle-type orbits. Later Valya Afraimovich informed Yakov G. Sinai about this new object of potential interest for the ergodic theory. After my talk at Sinai's seminar at Moscow State University, the map gained some prominence and became a subject of study in ergodic theory by young and well-established mathematicians, including Sinai's former student, Leonid Bunimovich. I also gave a talk at Anosov's seminar, and Dmitry V. Anosov's pupils (Yakov Pesin, Evgeny Sataev, and Nikolai Chernov) began their analysis of the ergodic properties such as the transitivity and invariant measures of multi-dimensional modifications of the Belykh map that laid the foundations for the "generalized hyperbolicity" theory [146, 165, 166]. At that time, I was working on my second doctoral thesis, and Anosov strongly recommended me to include the description of my map and its detailed analysis in the thesis. Later, in 1983, Anosov served as an opponent for my "Doctor of Sciences" degree thesis (see Figure 14).

Giving back to my alma mater, in 1977, I started teaching a graduate course on qualitative methods in nonlinear dynamics in Andronov's Chair of Oscillation Theory and Automatic Control. In particular, this course covered homoclinic orbits, Shilnikov's saddle-focus theorem, and the dynamics of the Lorenz system [24]. This would be the standard content for an advanced nonlinear dynamics course by modern standards; in 1977, it was quite rare. I vividly remember the first group of excellent students I taught, including Arkady Pikovsky, who later became a leading expert in coupled oscillator theory.

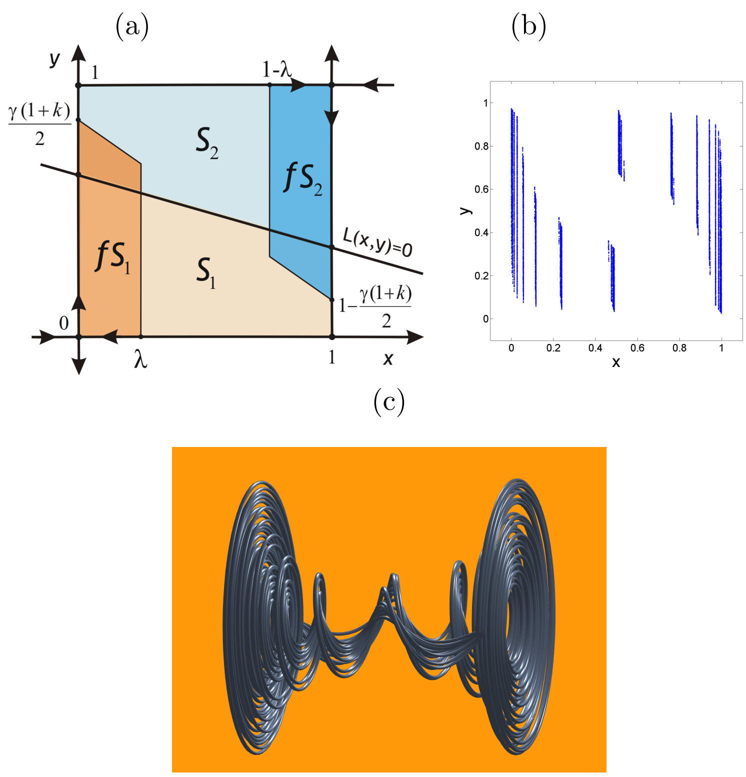


Figure 13. (a). First image of the unit square $S = S_1 \cup S_2$ under the map f (1)-(2), where S_1 and S_2 are separated by the discontinuity line $L(x, y) = 0$. One iterate of f transforms trapezoids S_1 (light brown) and S_2 (light blue) into fS_1 (dark brown) and fS_2 (dark blue), respectively. (b). Chaotic Belykh attractor generated by the map f with parameters $\lambda = 0.48$, $\gamma = 1.3$, and $k = 0.5$. (c). Embedding of the Belykh attractor into the 3-D phase space, see [29] for the corresponding 3D flow system.

Figure 14. Vladimir Belykh (left) and Dmitry Anosov (right) at a conference in Moscow.

In 1979, George Bruun, the Danish professor and my host during the 1975-1976 stay in Lyngby, invited me to spend a few months at DTU to continue our collaborative work. To remind the young reader, these were the times when most Soviet citizens were not allowed to travel abroad by the Soviet State, and an exit permit was required. Mine was delayed, and the entire trip to Denmark could quickly become elusive. To make it happen, George arranged for an invitation letter, signed by the Danish Queen, that opened all the bureaucratic doors and let me travel.

Figure 15. (From left to right): Vladimir Nekorkin, Valentin (Valya) Afraimovich, Lev Lerman, and Vladimir Belykh during Valya's short visit to NII PMK from Georgia Tech, one of the places he worked in the U.S.A. in the 90's. Photo circa 1996.

In 1980, I made a bet with Valya Afraimovich on a bottle of Cognac that I prove the existence of a homoclinic orbit in the genuine Lorenz system. Victor I. Yudovich, a professor from the University of Rostov-on-Don who one year before proved the existence of a homoclinic orbit in the Lorenz system, in a particular case of a small parameter, was our bet witness. It turned out that my original plan of attacking this problem was flawed. It took me about a month of trying and sleepless nights to find a solution and prove the statement. To fully convince Valya and other colleagues, I gave two long seminars to provide comprehensive, step-by-step proof of my theorem for the existence of a butterfly homoclinic linkage in the Lorenz system. The proof draft was published in [24] and the complete one in [26]. Valya lost the bet but was very happy with my success; we opened that bottle of Cognac together. Given the limited communication between the West and the Soviet Union, it took a while for this result to become known worldwide. Eventually, this theorem made its way into the state-of-the-art analysis of the Lorenz system and, in particular, was cited in the classical dynamical systems book by John Guckenheimer and Philip Holmes [79].

In 1983, I moved to the Institute of Water Transport Engineers (an engineering school in downtown Nizhny Novgorod, a five-minute walk from NII PMK). I took the position of its Mathematics Department's head, which became available after the retirement of Nickolai N. Bautin, Andronov's disciple, known in particular for the Bautin bifurcation and Bautin theorem [21]. Also, in 1983, I met Vadim Anischenko at a conference on the Oka River, with whom we quickly became life-long friends.

Figure 16. From left to right: Leonid Shilnikov, Diana and Leon Chua, Vladimir Belykh, Valery Ponomarenko, Lev Lerman, and Albert Morozov. Nizhny Novgorod, June 1992. The group picture was taken against the Research Institute for Applied Mathematics and Cybernetics (NII PMK) backdrop and a memorial plaque featuring A. Andronov's bas-relief.

Figure 17. Leonid Shilnikov (left) and Vladimir Belykh (right), 2005.

The Fall of the Iron Curtain intensified international collaborations and allowed easier exchange visits (see Figure 15). After meeting Leon Chua at a conference in Crimea, I started collaborating with him and spent a while at UC Berkeley working with Leon, Chai Wah Wu, and Ljupco Kocarev (see Figure 16). My research interests have broadened to the dynamics of networks and synchronization that later led to long-term collaborations with Erik Mosekilde, a physics professor at DTU, and Martin Hasler, an engineering professor at École Polytechnique Fédérale de Lausanne (EPFL). This important chapter of my scientific life deserves a separate description and will not be detailed in this essay.

I am 80 and still hold a professor position without immediate retirement plans. Reflecting upon my journey, I am profoundly grateful for the bonds I formed and the scientific camaraderie I shared with my late friends Leonid Shilnikov (see Figure 17), Valya Afraimovich, and Vadim Anishenko. Their enduring influence on my life and work remains a source of immense pride. Equally gratifying is the realization that my Ph.D. students have blossomed into accomplished and highly respected scientists who have left indelible marks in their respective scientific domains. Notably, Vladimir Nekorkin became a leader in nonlinear wave and coupled oscillator theory; Andrey Shilnikov, L.P.'s son, has emerged as an authority in global bifurcation theory; and Nickolai Verichev was among the pioneers in the discovery of chaos synchronization back in 1986. Witnessing their success, as well as the accomplishments of the disciples of my dear old friends, I believe it is fair to say that the future of dynamical systems is in good hands.

I extend my heartfelt gratitude to René Lozi, whose enthusiasm, encouragement, and determination made this essay possible. As much as the Lozi [117] and Belykh attractors share in common, René and I have a profound mutual respect and friendship, bound by our shared admiration for the renowned Russian singer and humanitarian, Bulat Okudzhava whose song we once sang together at the L.P. Shilnikov's memorial conference in Nizhny Novgorod in 2013.

5. Thrilled by nonequilibrium phase transitions by Eckerhard Schöll

As a school boy in a small German town, Nürtingen near Stuttgart in Southwest Germany, I was interested in a very broad spectrum of subjects, ranging from Latin, English, French, German via History, Music, Art to Mathematics and Physics. While I could not really decide whether my favorite subjects were languages, music, history, or mathematics, I was always fascinated by theoretical physics, since I did not really understand what it was about, but it sounded complicated and challenging. I loved repairing old clocks, building radios, and solving maths puzzles. But when I passed the final school exam (Abitur) with best grades in all subjects in 1970, it was clear that I wanted to study Physics at the University. During my undergraduate studies at the Universities of Stuttgart and Tübingen I discovered that I enormously enjoyed pushing the borders of my knowledge to challenging new horizons, in particular in mathematics and theoretical physics.

In a Physics Colloquium in Tübingen in 1974 I heard a lecture by Werner Heisenberg on his unified field theory of which I did not understand anything, but I was fascinated by his personality. In Mathematics seminars by Rainer Nagel, together with a group of fellow students, Eberhard Müller and Burkhard Kümmerer, I studied ergodic theory, functional analysis, von Neumann algebras and C^* algebras which were cutting-edge topics at that time. At the same time Prof. Harald Stumpf (1927 - 2021) became my first mentor by offering me a Master Thesis (called Diploma Thesis at that time) in theoretical semiconductor physics (calculation of generation-recombination processes of electrons). For this, besides analytics, I had to do computer simulations of the quantum mechanical transition probabilities and the rate equations, i.e., differential equations. Already in the summer vacations 1973, I had taken a programming course in Fortran at Imperial College of Science and Technology in London, besides language courses at Exeter and Broadstairs/UK. At that time, the code had to be punched onto cards, and they were fed manually as a batch into a central computer. For my Master

Thesis, I had to drive to the Computing Center of the University, punch my cards, and on the next day come back to collect the output on large sheets of paper, in most cases only to see that the code had not worked because of flaws. In Tübingen, a newly appointed professor, Werner Güttinger, introduced a course on nonlinear dynamical systems, where I made my first contact with this field. Probably the first conference which I attended when I was still an undergraduate student in Tübingen was the mathematically oriented "Rencontre entre mathématiciens et physiciens théoriciens" in Strasbourg in 1976, where I was very impressed by a talk by David Ruelle. In summary, I studied a wide selection of basic and special courses in theoretical and experimental physics and pure and applied mathematics, and moreover also history of arts, musicology, and philosophy, and all through my life, I played the piano and sang in choirs, performing many concerts.

After finishing my Diploma degree in physics in 1976, I wanted to continue my studies in England, since I had always been fascinated by foreign cultures and languages. I obtained a grant by the prestigious German Academic Scholarship Foundation (Studienstiftung des deutschen Volkes). In order to explore different possibilities, since I was still undecided whether I wanted to focus on semiconductor physics or mathematical physics of von Neumann algebras, I traveled to Britain and visited two scientists who represented quite different fields. When I first met Prof. Peter T. Landsberg in Southampton, he showed my his latest publication [106], and I was immediately thrilled by this, since it combined generation and recombination processes in semiconductor with nonequilibrium thermodynamics and a nonlinear dynamical systems approach. At this bifurcation point in my scientific life, I decided without hesitation to indulge into this project, and this determined my main field of research for the next twenty years, and influenced my scientific career up to the present day.

I started to work with Peter Landsberg (1922 - 2010) in Southampton, and he became a mentor and a friend to whom I owe a lot. I was accompanied by my girl-friend Viola, who later became my wife, and traveled with me to many scientific conferences and research visits throughout the world for 48 years until her much too early passing in 2023. Peter Landsberg, in his meticulous and ingenious way, introduced me to doing research and writing papers, and taught me attention to detail, which is equally essential for a researcher as having brilliant ideas. He has been a friend far beyond my Ph.D. times, and we continued to collaborate and visit each other until his death. Besides generation – recombination (g-r) processes of electrons and holes in semiconductors, a topic which he had pioneered [105] as well as the fundamentals of thermodynamics [104], he introduced me to semiconductor laser dynamics which subsequently and in parallel became my second topic [177]. Both cases were applications of nonlinear dynamics in terms of nonlinear ordinary differential equations. We also discussed difference equations (iterated maps), which represent a time-discrete version of the dynamics, and exhibit rich nonlinear dynamic scenarios including a route to chaos, in spite of their surprizingly simple form, e.g. the logistic map

$$x(n+1) = ax(n)(1 - x(n)), \tag{3}$$

where $x(t)$ is the dynamical variable and a is the bifurcation parameter. This resulted in a paper on applications to semiconductors much later [107]. As a small note, the value of the famous Feigenbaum constant, which gives the universal asymptotic ratio of successive period-doubling bifurcations in the logistic map, was found by Siegfried Grossmann and Stefan Thomae [78] already before it was published by Mitchell Feigenbaum [67]. At Southampton I attended also lectures by David Chillingworth who came

Figure 18. Photo taken at my graduation ceremony in 1978 at Southampton University, UK.

Figure 19. Photo taken at my graduation ceremony in 1978 at Southampton University, UK. From left to right: Eckehard Schöll, Peter Landsberg, Academic Registrar of Southampton University.

from the Warwick dynamical systems group and put this research into the more mathematical framework of bifurcation theory and catastrophy theory, and by David J. Wallace on renormalization group theory and critical phenomena which was an exotic but interesting new physical topic for me. As it turned out, I was able to complete a Ph.D. Thesis in Applied Mathematics in Southampton on "Nonequilibrium Phase Transitions in Semiconductors" in 1978 (see Figures 18, 19).

Our first published papers dealt with models for first and second order g-r induced non-equilibrium phase transitions in semiconductors [179], which were built on the analogy of the nonlinear rate equations with chemical rate equations, in particular the famous Schlögl model [167]:

$$\dot{x} = -x(x-a)(x-1). \tag{4}$$

I had previously made contact with Prof. Friedrich Schlögl (1917 - 2011) at the RWTH Aachen (Aachen Institute of Technology, Germany), and he had offered to me to do a Ph.D. under his supervision. He became my third mentor, and from him I have learned a lot about teaching physics and about physical thinking. After finishing my British Ph.D. in Mathematics at Southampton, I moved to Aachen and prepared another Doctoral Thesis (Dr. rer. nat.) in Physics, which I completed in 1981. It dealt also with the field of dynamical systems, but developed the generation - recombination models in a different direction. Studying a doped semiconductor, and considering impact ionization of electrons from ground state and excited state donors, I was able to develop a model which showed bistability in the S-shaped current-voltage characteristic (switching between the two stable branches is nowadays called tipping), with an unstable middle branch of negative differential conductivity (see Figure 20), and hence resulted in a plethora of self-organized spatial and spatio-temporal instabilities in the form of current filaments with a high conductivity core surrounded by low-conductivity [168, 169]. The field of self-organization and spatio-temporal pattern formation based upon nonlinear partial differential equations had become a focus of research at that time, and examples from laser physics, chemistry (Brusselator developed by Nicolis and Prigogine and collaborators, Belousov-Zhabotinsky reaction), and fluid dynamics (Taylor and Bénard instabilities) were abundant, but semiconductors had not been noted in this context except for a few works by Russian authors. In the following years in Aachen I explored and developed this field in great detail and in many aspects. Hermann Haken had created the field of Synergetics which deals with these nonlinear systems far from thermodynamic equilibrium and their universal features. In his pioneering work, he interpreted the laser transition as a nonequilibrium phase transition [77]. and discovered the analogy between the semiclassical laser equations and the Lorenz model in fluid dynamics, which gives rise to chaotic laser dynamics [82]. Concepts from thermodynamics and statistical physics have been applied to describe self-organization, spatio-temporal pattern formation, phase coexistence, critical phenomena, and first and second order nonequilibrium phase transitions [83]. Much more recently these ideas have been applied to networks of oscillators, where synchronization transitions may arise, giving birth to a plethora of partial synchronization patterns and complex collective behaviour, tipping transitions, explosive synchronization, nucleation, critical slowing down, etc., with applications to many natural, socio-economic, and technological systems [38, 176, 198]. A very recent application of that paradigm is heterogeneous nucleation of partially synchronized patterns in adaptive networks leading to two scenarios of first order nonequilibrium phase transitions [68].

Hermann Haken became very important in my scientific career when he invited me to Stuttgart in 1983 to give a seminar on my work on nonequilibrium phase transitions and self-organization in semiconductors. He was very interested in this recent addition to the family of Synergetics and invited me to write a monograph for his Springer Series in Synergetics. I happily agreed, and the volume was published in English in 1987 [170], and translated into Russian in 1991. The book has been cited more than 600 times. This influenced my whole direction of research for the coming years up to now. I submitted the book as my habilitation thesis at the RWTH Aachen, and after this became a Lecturer at RWTH Aachen and started my own group of students. In those years, I personally met many pioneers of nonlinear dynamical systems, and I initiated several collaborations with colleagues who performed experiments on nonlinear dynamics in semiconductors, which led to very fruitful joint workshops, research visits, and many joint publications, in particular with the Regensburg group of Wilhelm Prettl [205], the Tübingen group of Joachim Peinke and Jürgen Parisi [145, 180], the Japanese group of Kazunori Aoki [16], and later the Berlin group of Marion Asche [100].

In April 1989, a few months before the Berlin wall came down, I accepted an offer from the Technische Universität (TU) Berlin as Full Professor of Theoretical Physics, where I worked and taught for 30 years. During that time, I had the great chance to establish Nonlinear Dynamics, Complex Systems and their Control as a new internationally visible focus at TU Berlin. I organized many International Conferences on Control of Complex Systems and Networks (e.g., Palma de Mallorca 2012, Warnemünde 2014, Toronto 2015, Usedom 2016), and established collaborative research centers: I was Deputy Chairman and Principal Investigator of the Collaborative Research Centers on Semiconductor Nanostructures (SFB 296, 1994–2002), on Complex Nonlinear Processes (SFB 555, 1998–2010), and Founder and Chairman of SFB 910 on Control of Self-Organizing Nonlinear Systems (2011–2018). As a retired professor I am still active in research and in conferences worldwide, moreover I am Principal Investigator of the Bernstein Center for Computational Neuroscience Berlin (since 2010) and a guest scientist at the Potsdam Institute for Climate Impact Research (since 2020). Over the years I have been involved in many international collaborations; in 2000 I held a Fulbright Senior Scholar Award at Duke University, USA; in 2004 a Visiting Professorship of the London Mathematical Society; and in 2017 I received an Honorary Doctorate from Saratov State University, Russia, as a result of my very fruitful and active collaboration with the Saratov group of Vadim Anishchenko and Galina Strelkova. I am President of the International Physics and Control Society (IPACS, since 2019), a member of the German Physical Society (DPG), and a member of the Italian Society for Chaos and Complexity (SICC). Since 2021 I am Speciality Chief Editor of the new open access *Journal Frontiers in Network Physiology: Networks of Dynamical Systems.*

My first talks which I gave at International Conferences were in 1978 at the International Conference on the Physics of Semiconductors (ICPS) in Edinburgh, in 1980 at the German Physical Society (DPG) Annual Spring Meeting in Münster, and in 1981 at the International Conference on Hot Carriers in Semiconductors (HCIS) in Montpellier, France. All of these were parts of regular annual or biannual conference series, which I subsequently attended many more times. In particular, the DPG Spring Meetings have closely accompanied my life and my career for the last 45 years, since I first attended them as a student, later as a scientist giving talks and organizing Focus Sessions, and finally from 2008 to 2018 as the Local Organizer of the big Annual Spring Meetings in Berlin with over 6,000 participants each time. I always enjoyed meeting personally all those people whose names I knew from their published works, but now we had stimulating discussions, and often new ideas arose from those. At the conference in Montpellier I met Melvin P. Shaw from Wayne State University, Detroit, USA, whom I knew as author of famous papers and a book on semiconductor instabilities and in particular of a book on the Gunn effect. We arranged that I would visit him in the US for a year at the Department of Electrical Engineering, and as a result of this year which I spent with him in 1983/84 as a Visiting Assistant Professor, I became a coauthor of the book *The Physics of Instabilities in Solid State Electron Devices* which was finally published in 1992 [188].

In 1986 I attended for the first time Dynamics Days in Enschede, Netherlands, a conference series which was founded by Robert Helleman in 1980, and which is organized annually up to the present day. The first seven meetings were held at Twente University, Enschede, as very small informal workshops, then they shifted to Düsseldorf under the chair of Gert Eilenberger for four years, and afterward moved all around Europe. Dynamics Days Europe is the oldest Conference in the

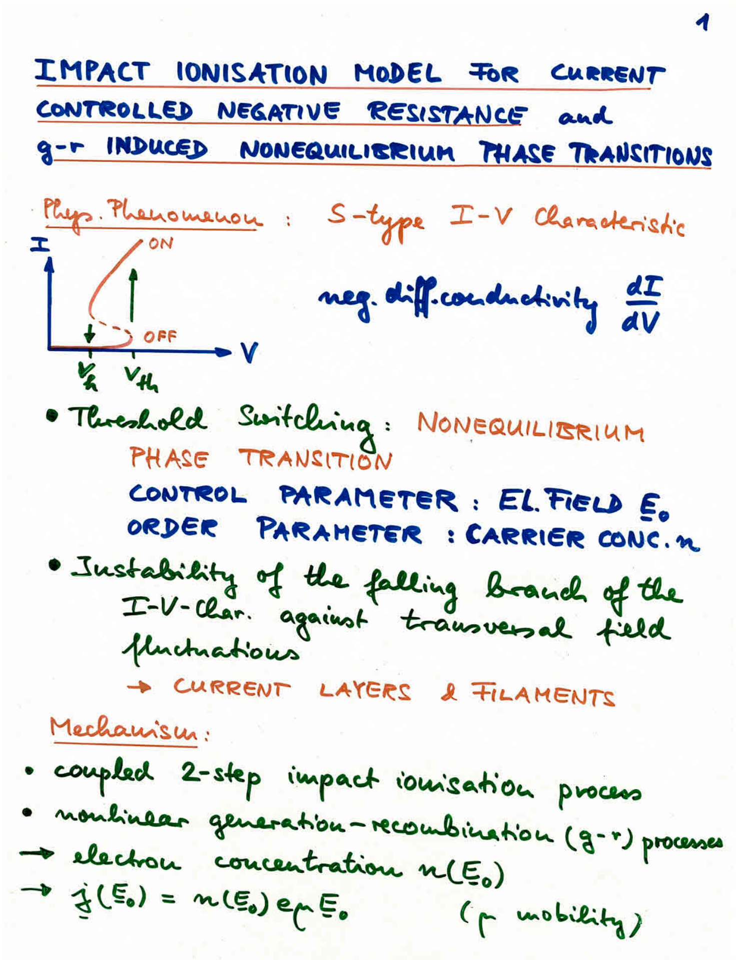


Figure 20. First slide of my talk at the International Conference on Hot Carriers in Semiconductors in 1981. At that time transparencies were written by hand and projected by an overhead projector.

Figure 21. Dynamics Days Europe in Berlin 2005. The photo was taken at the conference excursion on the river Spree. In front: Bernd Krauskopf (left), Eckehard Schöll (right).

Dynamics Days family, which now comprises Dynamics Days Europe, Dynamics Days US, Dynamics Days Asia Pacific, Dynamics Days Latin America and the Carribean, Dynamics Days Central Asia.

Over the years it became my main conference, and I have attended 23 editions of this series, organized one myself in Berlin in 2005 (see Figure 21), and have been a member of the International Advisory Committee since 2005, and its chair 2016-2019. As the focus of this conference over the years shifted from chaos, self-organization and nonequilibrium phase transitions to a wide spectrum of topics in mathematics, physics, life sciences, and engineering ranging from networks to machine learning, brain dynamics, power grids, climate modelling, also my own focus of research has been expanding and developing over the decades. Often this shift has been fertilized by the interaction with other researchers on conferences and workshops. After extending nonlinear dynamics to semiconductor nanostructures [171, 172] and quantum wires, quantum dots, quantum transport, growth kinetics of quantum dots and cell populations, electro-optical nonlinearities, nonlinear laser dynamics, I studied more general delayed complex systems and networks [71, 92], chaos control [181] (the Handbook on Chaos Control has been cited more than 1.000 times) and in particular time-delayed feedback control [178], synchronization on delay-coupled networks [173, 174], partial synchronization patterns [176], chimera states [175, 182, 208] which are intriguing hybrid states where the network spontaneously splits into coexisting synchronized (coherent) and desynchronized (incoherent) domains, and applied these phenomena to brain dynamics and power grids. The methods involved both ordinary and partial differential equations and difference equations, and deterministic and stochastic effects. As an example, if the simple logistic map Eq.4 is coupled nonlocally in a ring network [143], intriguing chimera states arise:

$$z_i^{t+1} = f\left(z_i^t\right) + \frac{\sigma}{2P} \sum_{j=i-P}^{i+P} \left[f\left(z_j^t\right) - f\left(z_i^t\right)\right], \quad (5)$$

where z_i are real dynamic variables ($i = 1, ..., N$, $N \gg 1$ and the index i is periodic mod N), t denotes the discrete time, σ is the coupling strength, P specifies the number of neighbors in each direction coupled with the i-th element, and $f(z)$ is a local one-dimensional map. We choose f as the logistic map $f(z) = az(1-z)$ and fix the bifurcation parameter a at the value $a = 3.8$. This choice yields chaotic behaviour of the map f.

A typical scenario of the coherence-incoherence transition via chimera states is illustrated in Figure 22(a)-(f), where we fix the coupling radius $P/N = 0.32$ and decrease the coupling strength σ. First, in Figure 22(a), the solution profile z_i^t is clearly smooth for $\sigma = 0.43$. Thus, the network dynamics is spatially coherent. For smaller σ, the profile z_i^t sharpens up and, at some value $\sigma \cong 0.40$, loses smoothness in two points x_1 and x_2 as shown in Figure 22(b). This is a bifurcation point for the coherence-incoherence transition: beyond this parameter value, the wave-like profile z_i^t splits up into upper and lower branches, and two narrow boundary layers of incoherence are born around the points x_1 and x_2 (shaded yellow stripes α_1 and α_2 in Figure 22(c)): a chimera state. The incoherence stripes become wider with further decrease of σ (Figure 22(d)) and, eventually, the dynamics becomes completely incoherent (Figures 22 (e) and (f)).

Looking back, I have been very fortunate since I have been able to realize a lot of the goals which I set to myself during the past 50 years since I started studying physics. What are the essentials in life? In the center of my professional life have always been the young people to whom I gave guidance, whom I helped to make their first steps in science, and develop their own carrier. I have supervised over 150 Diplom, Master, and Bachelor Theses, 36 Ph.D.s, and 4 Habilitations. This is my secret: since I have spent most time with these young people, I have also stayed young! And there is an essential point: science, after all, is made by humans, and it is essential to stay human in this sometimes rough world.

Another essential thing in my life: Science is very international. I am happy to be part of this international family of physicists and mathematicians. This starts with student exchanges with foreign countries which I have always supported and encouraged. Then, international collaborations with colleagues all over the globe, and conference all over the world, where this family meets, form a network. Having lived in England and in the US for several years, and in different regions in Germany, and having very active collaborations with Russia and many other countries, I feel at home in many places of this world, and in my heart I have taken something from every place.

Finally, my thanks go to all the people who have stimulated and enriched my scientific life by their contributions as students, collaborators, and mentors. As Isaac Newton said: If I have seen further it is by standing on the shoulders of giants.

6. Impact of nonlinear dynamics on my scientific life by Galina Strelkova

During my high school years, my interests were quite broad. On the one hand, I really liked the natural sciences, such as mathematics and physics. These were my favorite subjects in school and I was ready to give them all my time. On the other hand, I was

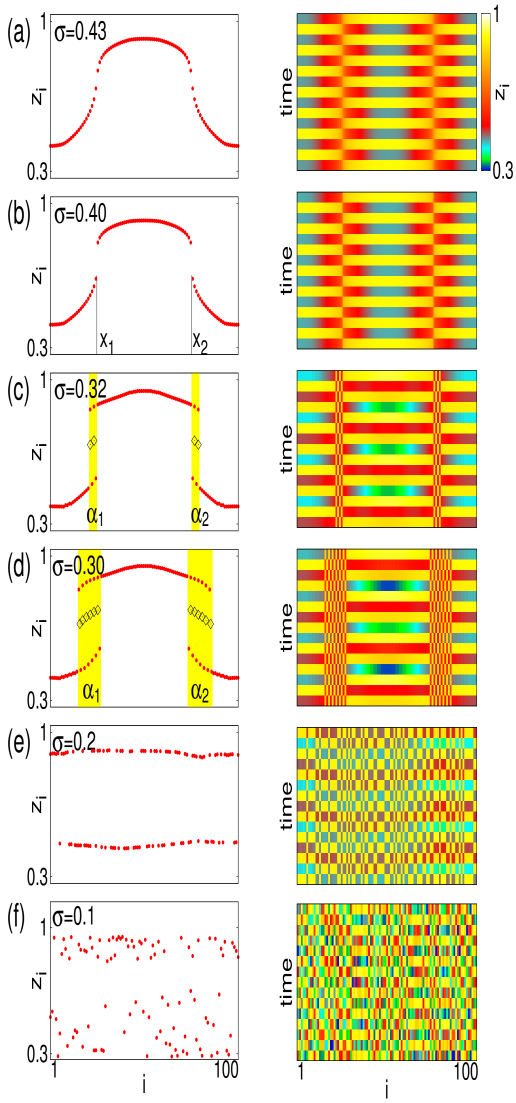


Figure 22. (Color online) Coherence-incoherence bifurcation for coupled chaotic logistic maps for fixed coupling radius $P/N = 0.32$. For each value of the coupling parameter σ (decreasing from top to bottom, $\sigma = 0.43, 0.4, 0.32, 0.3, 0.2$, and 0.1, respectively) snapshots (left columns) and space-time plots (right columns) are shown.

very attracted to medicine. I read books on surgery with great interest and sometimes I was even allowed to attend operations in clinics (my mother was an anesthesiologist and resuscitator). In addition, over all my school years I also seriously studied music (piano and vocals), composed music for poetry, took part in various competitions and became their laureate several times. It was expected that after graduating from school I would certainly continue my studies either at a medical institute or at the Conservatory. However, by the time I graduated from high school, I definitely decided for myself to become a school teacher in mathematics, physics and computer science. The last subject was so curious for me (although there were no computers at school yet) that I just dreamed about it. I was incredibly fond of mathematics, but to a greater extent not as a beautiful and abstract science, but as an essential tool for describing natural phenomena and events and solving the most interesting physical problems.

I lived in a small city and after graduating from high school I went to Saratov (a big regional center on the Volga river, 260 km from my native place) to enter the Pedagogical Institute. Since I graduated from high school with honors, I was advised to continue my studies at the Department of Radiophysics of the Faculty of Physics of Saratov State University. I entered there without exams and since then my entire conscious (professional and scientific) life has been inextricably linked with the Department of Radiophysics.

My first introduction in the field of dynamical systems came when I was a first-year physics student at the Department of Radiophysics (since 1995 it is the Radiophysics and Nonlinear Dynamics Department) of Saratov State University. This was in the early 90s. The Department was headed by Professor Vadim Anishchenko (1943-2020) who was also a scientific supervisor of a just organized Nonlinear Dynamics Laboratory. Stochastic dynamics or deterministic chaos was the key direction of research of this group. At the end of the first course I had to prepare and present a course work that was dedicated to the transition to chaos in the Hénon map [85]:

$$\begin{cases} x_{n+1} = 1 - a x_n^2 + y_n, \\ y_{n+1} = b x_n. \end{cases} \tag{6}$$

I knew nothing about this system and chaos at all and that was my first own investigation and attempt to figure out in this issue and to perform numerical analysis. My high school dream came true! I got the opportunity to learn how to use a computer and conduct numerical simulations. Of course, at that time computer technology was only developing in our country, but the Nonlinear Dynamics Laboratory had its own personal computers. In the future, thanks to the cooperation and support of Professors Jürgen Kurths and Leon Chua, as well as to grants from International and Russian research foundations, our computer park was constantly expanded and strengthened.

After listening the lecture course on deterministic chaos given by Professor Anishchenko and reading a number of papers and reviews [85, 152, 158] some aspects in the topic of deterministic chaos became clear and I had decided to continue studying in this research direction. Moreover, I was impressed by the fact that a simple model system in the form of difference equations can demonstrate a nontrivial and, in some sense, exotic behaviour such as chaos, and the main properties and important regularities established for the maps can also be observed in more complicated systems described by ordinary differential equations. In particular, it was shown that the Hénon map is similar to a map generated in the Poincaré section of three-dimensional systems, such as the Rössler oscillator and the Anishchenko-Astakhov oscillator (the latter is a real radiophysical device) [14].

Going deeper into the study of the topic of dynamical chaos and its mathematical image - a strange attractor [158], the understanding came that chaotic self-sustained oscillations can be significantly different in their properties and thus, there are differences in the structures of the associated attractors. Strange attractors are robust hyperbolic objects [158, 191] and satisfy a number of rigorous conditions [6]. These attractors are rather "ideal" and only few mathematical models and physical devices can realize robust hyperbolic attractors [102, 148, 206]. However, real systems and devices typically demonstrate chaotic self-sustained oscillators which correspond to *quasihyperbolic* and *nonhyperbolic* attractors [191]. The first group of chaotic attractors include quasihyperbolic Lozi, Belykh, and Lorenz-type attractors [25, 113, 116, 146]. Nonhyperbolic attractors are realized in many dynamical systems, i.e., the Hénon map, the cubic map, the logistic map, the Rössler system, the Anishchenko-Astakhov oscillator, the Chua's oscillator etc.

When I graduated from the University, Professor Vadim Anishchenko invited me to continue my postgraduate studies under his supervision and at the same time to work as a research engineer at the Department. Of course, I agreed with great pleasure! Since then, I also became a member of the Nonlinear Dynamics Laboratory. The Lab also included a few Associate Professors who defended later their Doctor of Sciences (Habilitation) theses (Vladimir Astakhov, Tatiana Vadivasova, Alexander Neiman, Dmitry Postnov, Alexey Shabunin) and became Professors at the same Department or moved to another University. Several Ph.D. students were also involved in the research work in the Nonlinear Dynamics Laboratory and defended their Ph.D. theses (Olga Sosnovtseva, Natalia Janson, Alexey Pavlov, Alexander Balanov, Igor Khovanov, Alexander Silchenko, Alexander Nikitin). Many of them now are recognized experts in the nonlinear dynamics and statistical physics field and have permanent positions in various European and American universities and research centers.

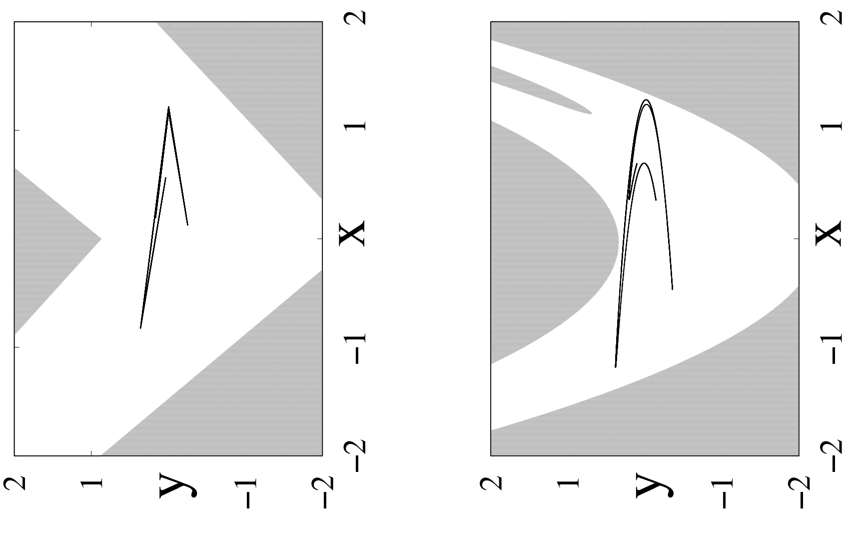
In order to classify and understand the properties of chaotic attractors from a physical point of view, Professor Vadim Anishchenko suggested me to study this issue in the framework of my Ph.D. thesis. Exploring the dynamics of the aforementioned systems by using different dynamical and statistical characteristics and measures, I was able to define the fundamental differences in the properties of quasihyperbolic and nonhyperbolic attractors [11]. The main model systems under study were the Lozi map (Figure 23(a)):

$$\begin{cases} x(n+1) = 1 - \alpha\,|x(n)| + y(n), \\ y(n+1) = \beta x(n). \end{cases} \tag{7}$$

and the Hénon map (6) (Figure 23(b)). I successfully defended my Ph.D. thesis in 1998 at Saratov State University. The results obtained were then highlighted in lectures on the theory of discrete-time systems and dynamical chaos and were also included into the book [14] published in Springer in 2014. As it turned out later, the results obtained on the classification of chaotic attractors were very useful in further research.

After defending my Ph.D. thesis I was invited several times by Professor Jürgen Kurths for research visits in his Nonlinear Dynamics Working Group in Potsdam University. It was a great experience and we published several joint papers (for example, [10, 12, 13]). The successful cooperation with Professor Kurths is still continuing and this is very important for us.

As I mentioned above, my professional career was inextricably linked with the De-



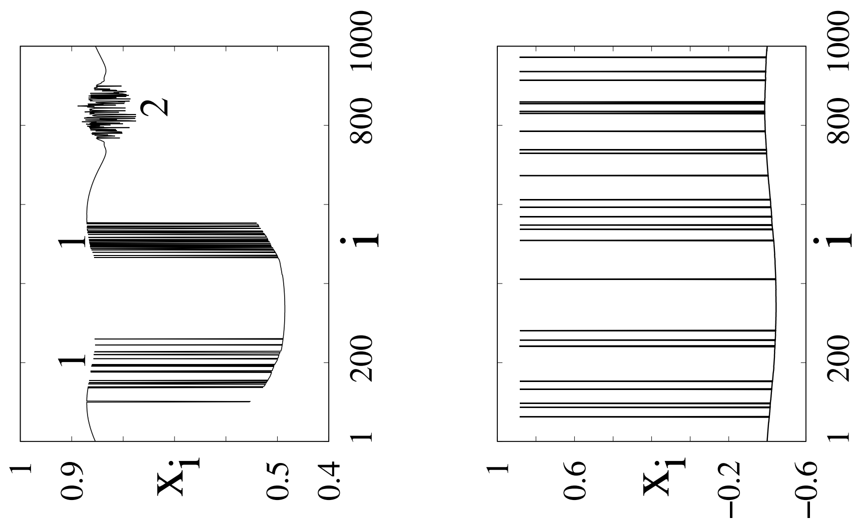
(a) (b)

Figure 23. Chaotic attractors (black curves) and basins of their attraction (white region) (a) in the Lozi map (7) for $\alpha = 1.5$ and $\beta = 0.3$ and (b) in the Hénon map (6) for $\alpha = 1.32$ and $\beta = 0.3$. Trajectories from the gray region go to infinity.

partment of Radiophysics and Nonlinear Dynamics. Around 2002-2003, I moved to assistant professor, and in 2010, I was promoted to associate professor in the same department.

At the beginning of 2015, some very important and pleasant changes happened in the scientific life of our group. Thanks to the initiated collaboration and the activities of Prof. Eckehard Schöll, Technical University of Berlin, our group was involved in Collaborative Research Center (CRC) SFB 910 (2011-2022): Control of Self-Organizing Nonlinear Systems, in the framework of the first Russian Project in a German CRC. Professors Vadim Anishchenko and Tatiana Vadivasova were Principal Investigators (PIs) of the Project and in 2019 I became a PI too.

The scientific cooperation with Prof. Eckehard Schöll and Dr. Anna Zakharova turned out to be very fruitful and successful. As a result, a new research direction appeared in our department, which was devoted to the studies of complex spatio-temporal structures, such as chimera states [101], in networks of coupled nonlinear oscillators. I remember very well with what enthusiasm and interest we searched and studied the relevant literature, how we organized and performed our own research on chimera states, how enthusiastically we discussed the results obtained and made joint research plans with our colleagues from the Technical University of Berlin during our numerous scientific visits. And here our knowledge on the peculiarities of dynamical systems played a key role. The first bright idea was expressed by Professor Vadim Anishchenko, namely whether chimera states always accompany a transition from complete synchronization (coherence) to spatio-temporal chaos (incoherence) independently on the type of individual oscillators in a network. Before, these states were found numerically in networks of nonlocally coupled chaotic logistic and Hénon maps (Figure 24(a)) [143, 144]. However, our further studies showed that the ring network of nonlocally coupled Lozi maps demonstrates the transition from complete



(a) (b)

Figure 24. (a) Phase (1) and amplitude (2) chimeras in the network of nonlocally coupled logistic maps; (b) Solitary states in the network nonlocally coupled Lozi maps.

coherence to incoherence through the emergence of solitary states (Figure 24(b)) when the coupling strength between the elements decreases [185].

Working on this problem was very interesting and exciting. The members of the Nonlinear Dynamics Laboratory had an excellent opportunity to have research visits to the Technical University of Berlin, to give talks at scientific seminars of the groups supervised by Prof. Eckehard Schöll and Dr. Anna Zakharova and participate in various international conferences (see Figure 25). Together with our Berlin collaborators, we revealed the mechanisms of formation of amplitude and phase chimeras in networks of chaotic oscillators [40], explored the possibilities of external, mutual and relay synchronization of chimera and solitary states in multilayer networks of various coupled nonlinear dynamical systems [162, 163], as well as the stability of chimera and solitary states towards noise excitations [184]. In particularly, it has been shown for the first time that repulsive interlayer coupling can induce anti-phase synchronization of spiral and target wave structures, including chimeras, in networks of self-sustained oscillatory and neural networks [189]. It has also been shown that solitary states and solitary state chimeras can be observed in neural networks [161, 164].

At the beginning of 2019, I had a serious conversation with Professor Vadim Anishchenko, in which he suggested and, moreover, strongly recommended that I start preparing a Doctor of Sciences (Habilitation) thesis on this topic based on joint scientific publications. By that time, indeed, a large number of new and important results had been obtained, which formed a coherent and logical picture of serious scientific work [195]. I immediately followed my mentor's advice. In December 2019, my thesis work "Chimera structures in ensembles of nonlocally coupled chaotic oscillators" was accepted by the Dissertation council of Saratov State University for defense. After being rescheduled twice due to severe restrictions caused by the coronavirus pandemic, on October 1, 2020, my defense took place and was very successful. As Professor Vadim Anishchenko noted, my work was the first Doctor of Sciences thesis defended

Figure 25. At the conference of the German Physical Society, Regensburg, April 2019. In the front row, Prof. Anna Zakharova is first on the right. In the second row, Prof. Vadim Anishchenko is first on the right, Prof. Eckehard Schöll is third on the right, Dr. Galina Strelkova is fourth on the right.

in Russia, on the topic of special spatio-temporal structures in ensembles of interacting oscillators.

After the death of Prof. Vadim Anishchenko on November 30, 2020, I headed the Radiophyics and Nonlinear Dynamics Department of the Institute of Physics of Saratov State University, and our group is now successfully continuing to work in this direction in close and fruitful cooperation with our German colleagues.

In conclusion, I would note the *butterfly effect* which reflects a strong sensitivity of chaotic systems to initial conditions and which played a significant role in defining the direction in my scientific life.

7. My life to promoting difference equations by Saber Elaydi

Born in the sun-scorched land of Beer Sheba, Palestine, I emerged from humble beginnings that would shape my indomitable spirit. My family was ethnically cleansed in 1948 and escaped to a refugee camp in the Gaza strip, where I was raised amidst the trials and tribulations of a refugee camp. From a young age, I displayed an insatiable thirst for knowledge that transcended the confines of my circumstances. In a world plagued by uncertainty and limited resources, I sought solace and inspiration within the realm of mathematics. The numbers and theorems became my refuge, offering a sanctuary where possibilities knew no boundaries.

With limited access to formal education, my intellectual journey began as a self-taught endeavor. Armed with unwavering determination and an unquenchable thirst for learning, I delved into the world of mathematics, voraciously consuming every piece of knowledge I could find. Against all odds, my perseverance caught the attention of mentors and benefactors who recognized my talent. This unwavering dedication earned me a scholarship that paved the way for formal education, propelling me beyond the confines of the refugee camp. Embarking on a path of enlightenment, I pursued higher education with an unyielding resolve. And in 1964, I received a Bachelor's degree in mathematics and physics from Ain-Shams University in Cairo, Egypt.

Finally, in 1978, I received my Ph.D. from the University of Missouri in the US, under the guidance of Ping-Fun Lam and David Carlson. My dissertation, titled "preferred sets in topological dynamics", was focused on transformation groups. After my graduation, I was hired as an assistant professor at Kuwait University. There, I worked in three different projects with three different researchers. The first was with H. Farran on topics such as "On weak isometrics and their embeddings in flows, Isometrics and certain dynamical systems". The second was with F. Dannan on topics such as "Lipschitz stability of nonlinear systems of differential equations I and II". The third was with S. Kaul on topics such as "Semiflows with global extensions I and II, Notions of negative stability, Stability of limit and prolongation sets in semiflows". And in 1982, Otomar Hajek, from Case Western University, Cleveland, Ohio, visited Kuwait University to give a series of talks "on discrete and continuous semi-dynamical systems". I invited him with friends to celebrate the New Year's eve in my house. We all got drunk that evening drinking so much Scottish Whisky. He was very surprised, since alcohol is prohibited in Kuwait, and told me that he never had so much Whisky in his entire life. Two days later when we all got sober, I invited him to the faculty club for coffee and snack. Suddenly, he looked at me and said in a very serious tone "Saber, you should get out of this rotten place". I said "I agree with you and I have been thinking about this for a year". Two months later, I received an email from the department's chair in his University offering me a visiting appointment at Case Western Reserve University, and I accepted. Then in the summer of 1983, I moved with my wife and three children to Cleveland, Ohio. Hajek became my third mentor after my two doctoral advisors. My joint work with him was devoted to the study of dichotomy and trichotomy of nonlinear differential equations, two of the most important asymptotic properties of dynamical systems. In this direction, we published the papers: "Exponential trichotomy of differential systems, Exponential dichotomy" and "Trichotomy of nonlinear differential equations". Several years later, these results were followed by my joint work with K. Janglajew, University of Bialystok, Poland, "Dichotomy and trichotomy of difference equations".

However, I was eager to return closer to my original interest. And in 1987, I found it in difference equations and discrete dynamical systems. Here we have an action of a group, the integers, on Euclidean, metric, or topological spaces. During a conference in Orlando, I participated in a special session on difference equations, where I felt that difference equations play second fiddle to differential equations. That was the moment when I decided to change that situation and bring respect to the area of difference equations. The first step I took, with the help of my friend Gerry Ladas, was to establish the Journal of Difference Equations and Applications (JDEA) in 1994. The second step was to hold the first international conference (ICDEA I) on difference equations in San Antonio, Texas, in May 1994. ICDEA became a great success and has been an annual conference ever since, held on almost all continents (see Figures 26, 27).

At ICDEA I, I met two people that have had a great influence on my research and career, Bernd Aulbach of the University of Augsburg and Jim Cushing of the University of Arizona. Bernd invited me to his University to give a colloquium talk and to do joint research. As a result we published a paper, with his colleague Zeigler, titled "Asymptotic solutions of the Schrodinger equation arising from a Dirac equation with random mass". Then he raised a question whether or not a converse of Sharkovsky's theorem holds true which led me to publish a proof of it later. Since then and until his death in 2003, we became very close friends. And in 2001, at the ICDEA meeting in Augsburg, Bernd and I discussed the formation of a new society which we called "The international Society of Difference Equations (ISDE)". At that meeting, Bernd was elected president and I was elected vice-president (see Figures 28, 29).

Jim Cushing has greatly influenced my decision to work on a new area, namely, mathematical biology. Through his work and personal discussion, I started slowly but surely switching my research interest to mathematical biology, particularly the area of ecology. I met Jim the first time in 1994, when he attended ICDEA I in San Antonio. During my visit to the University of Southern California, I met with Robert Sacker (the author of the famed Neimark-Sacker bifurcation). At that time, he quit doing research in mathematics as his main area was ordinary differential equations, which was replaced by partial differential equations. I presented to him the Cushing-Henson conjecture that deals with non-autonomous periodic Beverton-Holt model in ecology. This was a turning point for me as we started working jointly and laid the foundation of nonautonomous periodic difference equations using the technique of skew-product discrete dynamical systems. Then I continued working in the area of mathematical biology focusing on developing the mathematical foundation of various open problems in ecology. And at the joint math meeting on January 13^{th}, 2006, I had a dinner with Jim Cushing at the River Walk in San Antonio, Texas. After a few beers, I suggested to Jim to start a new journal titled "Journal of Biological Dynamics" with emphasis on ecology and epidemiology. After some hesitation, Jim agreed to jointly submit a proposal to a publisher that I been working with through Journal of Difference Equations and Applications, Taylor and Francis. Our proposal was approved and we started the journal. This led to the initiation of a new conference titled "International Conference on Mathematical Modeling and Analysis of Populations in Biological Systems".

Two more people have had great influence on my research. The first was Lord Robert May. In the spring of 1995, I suggested to my (former) dean, John Dickey to invite him to give a talk as part of the "Distinguished Science Lectures Series" at Trinity University. He gave a lecture titled "How Many Species Are There?". It was a huge success with over 1500 in attendance. The day after the conference, I had the pleasure of asking him several questions about chaos and the evolution of species. And at dinner, I discussed with him my new book project on discrete dynamical systems. He liked the title "Discrete Chaos" but he asked me to add "with applications in science and engineering" to distinguish it from Devaney's book. Then he, graciously, suggested some valuable criticism and suggestions and promised to write the preface of the book, which he did. The second person is Jim Yorke. I met Jim Yorke at the fifth ICDEA, which was held in Temuco, Chili, in January 2000. I worked hard

Figure 26. ICDEA 2002, Changsha (China). Front row, fourth from the right: Bernd Aulbach, in the middle Saber Elaydi (with a white shirt, and no tie.)

with Gerry Ladas to convince Yorke to travel to Chili since he was worried about traveling overseas at that time. At that meeting, Jim gave a very convincing argument that difference equations is more general than differential equations, contrary to the prevailing conception. Since that time we have developed a lasting friendship exchanging ideas and engaging in discussions relevant to our common interests.

Throughout my career, I mentored numerous Ph.D. students and authored various publications, including *Discrete Chaos: Applications in Science and Engineering*, *An Introduction to Difference Equations*, *Upside-Down: The Interplay between Life and Chaos*, and co-authored with Jim Cushing *Discrete Mathematical Models in Population Biology: Ecological, Epidemic, and Evolutionary Dynamics*, among others.

While my mathematical achievements have been recognized, I never forgot my roots. With the help of my friend, Ulrich Eckern, University of Augsburg, we helped organize biennial conferences in mathematics and physics at universities in the occupied West Bank.

8. My Road to Dynamics by Jim Michael Cushing

I was raised in Cheyenne, Wyoming, in the western part of the United States. Despite being a small town in a large and sparely populated state, Cheyenne had an outstanding K-12 education system. Its broad curriculum offerings included courses in all the basic sciences and mathematics, up through calculus, all of which I took while a student there in the late 1950's. I was so enamored with mathematics that I would check out the text book from the local Carnegie library during the summer recess before I took a math course in order to teach myself, as best I could, the topic prior to the upcoming school year. In this way I was introduced to not only the standard courses in algebra, but plane geometry (using Euclid's Elements), solid geometry, trigonometry, and single variable calculus. During summer school recesses

Figure 27. ICDEA 2007, Lisbon. Front row, fifth from the right: Michał Misiurewicz, sixth from the left: Saber Elaydi. Second row third from the left Bernd Aulbach.

Figure 28. ICDEA 2000. From left to right Gerry Ladas, Bernd Aulbach with a beard, last on the right Saber Elaydi.

Figure 29. ICDEA 2001, Augsburg, Germany, July 30 – August 3, 2001. Front row, fifth from the right: Bernd Aulbach, to his right Alexander Sharkovsky, to his right Saber Elaydi (with a cap). On the top line in the middle, Gerry Ladas (to the left of the woman).

I would also occupy myself with applications of mathematics (measuring heights of trees using trigonometry, etc.), which was the beginning of a life long love of applied mathematics. I would add that, given the lack of activities available to young people in such a small town, I also spent a great deal of time studying piano performance, practicing sports (I played school basketball, football, and baseball teams), and reading philosophy (a lot of Aristotle).

Upon graduation from high school in 1960 I had various opportunities that included a scholarship offer in piano performance and basketball (of all things!). However, I felt I needed to ground myself more before making any heavy career decisions, so I left for a freshman year of college at the University of Oklahoma, where I had a good friend. There I skipped over introductory calculus and focused on courses in physics, chemistry, and geology. I transferred to the University of Colorado in Boulder for the last three years of my undergraduate education where I continued with courses in these sciences as well as astronomy and psychology. I needed to declare a major at this point and so I reflected back on my school subjects to see what it was I most enjoyed doing academically, with the result that I declared myself a mathematics major. I finished the major requirements for my bachelors degree before finishing my senior year, during which I enrolled in postgraduate mathematics classes. During that year I also participated, as part of the University's team, in the preeminent national William Lowell Putnam Mathematical Competition for undergraduate students (our team placed in the top ten). While at the University of Colorado I also continued my piano performance studies, which included coaching from the celebrated concert pianist Maria Clodes who was visiting the University at that time.

After graduation from the University of Colorado (1964), I was accepted for graduate school in mathematics at the University of California at Berkeley. However, at the last minute I received a phone call from the assistant director of the Institute of Fluid Dynamics and Applied Mathematics at the University of Maryland who offered me a National Defense Act Fellowship that would fund my graduate studies. At the time UC Berkeley didn't offer financial support to first year students and I had little financial resources available to me. In order to get through my undergraduate studies, I had worked several jobs ... from a handy man in a women's dormitory to a upper-class student advisor in a men's dormitory. So I turned my beat-up car from west to east and drove to College Park, Maryland, where I became a graduate student in applied mathematics.

My studies at Maryland, for the first year, were solidly concentrated in pure mathematics, as was the fad those days, both in analysis and algebra. (In my oral Ph.D. candidacy examination I remember being asked to outline the proof of the unsolvability of quintics by using Galois field extensions.) It was after passing the written and oral candidacy examination, that I got heavily into dynamics. My studies ranged from ordinary and partial differential equations to abstract dynamical systems and from basic theoretical matters (e.g. existence and uniqueness of weak solutions and distributions) to practical numerical analysis and coding (computers were just coming into use).

My Ph.D. dissertation advisor was Monroe H. Martin, who at the time was the director of the Institute of Fluid Dynamics (and former chair of the Mathematics Department at Maryland). He took on dissertation students only one at a time (one of whom, before me, was Simon A. Levin, who is currently the director of the Center for BioComplexity at Princeton University). Several former students of Professor Martin's, as well as I, worked on a conjecture of T. Levi-Civita concerning deep water waves and the uniqueness of waves corresponding to a certain physical parameter involving the wave length, the square of the wave profile's speed, and the gravitational constant. None of us settled the conjecture in our dissertations, but each made various contributions towards its resolution. My years at Maryland (1964-1968) were mathematically quite exciting and fruitful, in large part because of numerous high level faculty members and student classmates (one of whom was the celebrated James Yorke, who went on to coin the word "chaos" in dynamical systems).

Upon completion of my Ph.D., I celebrated by taking a summer off to roam around Europe (with friends in the counter culture remnants of the beatnik generation), from Scotland to Italy and Spain, before taking up a position as assistant professor of mathematics at the University of Arizona in Tucson in the Fall of 1968. On the faculty there at the time was the celebrated fluid dynamicist L. M. Milne-Thomson and some of his students. Milne-Thomson had written several books out of which I had studied (including *Theoretical Hydrodynamics and Theoretical Aerodynamics*) and I was thrilled to meet him. Although the teaching load was quite heavy, by today's standards, research publication was still expected and so I remained engrossed in Levi-Civita's conjecture until I settled it in the affirmative, for sufficiently small amplitude waves, in my first publication after that of my dissertation [48].

In 1970 Milne-Thomson moved on from Arizona and that same year I took a post-doctoral position at the IBM Thomas J. Watson Research Center, just north of New York City in Yorktown Heights. There I continued research on existence and uniqueness theorems for nonlinear elliptic partial differential equations, as they depend on model parameters, which naturally led me into a study of bifurcation theoretic methods. I also attained a renewed interest in dynamic stability while at IBM, mainly through stimulating discussions with Charles Conley (of Conley index fame), who was my office mate for a while.

Upon return to the University of Arizona, I began a collaboration with John Bownds, a colleague in the mathematics department, on stability theory for integral and integro-differential equations. We found interesting applications of our work in theoretical ecology, which for me began a career long adventure in mathematical biology and theoretical ecology. Integro-differential versions of the famous Lotka-Volterra models for population dynamics and species interactions appear in Volterra's original work [183]. As models, these types of equations (which were the topic of my first book [49]) often result from time delays related to demographic structure in a population. There are two broad types of models used for the dynamics of demographic structure, one utilizing continuous time and structure and the other utilizing discrete time and structure. Models of the first type are described by a first order hyperbolic partial differential equation (the von Forester equation) under nonlinear, nonlocal boundary conditions, on which I focussed my research activity in the 1970's. Such equations typically involve highly technical mathematical issues with regard to even the basic questions of existence of solutions and, because of this, can provide difficulties for theoretical biologists whose interests lie in other directions (ecological questions involving asymptotic dynamics, stability, bifurcations, etc.).

The second type of structured population models are the discrete time matrix models originally popularized by Lewis and Leslie in the 1940's. These models are described by systems of difference equations and do not entail mathematical difficulties surrounding the well-posedness of initial value problems. An introduction to both types of structured population dynamic models can be found in my second book [50], which resulted from a series of lectures I gave at North Carolina University in 1997, sponsored by the Conference Board of the Mathematical Sciences.

The period of my professional life, from the late 1970's to the early 1990's, was an active one that involved many studies and publications in theoretical ecology in conjunction with several postdoctoral students (Steve Simmes, Mohamed Saleem, Jia Li, and Yicang Zhou) and several graduate students (Zakaria Alawneh, Kathleen Crowe, Guillermo Uribe, Mona Alameddine-Roddier, BingXu, Utith Inprasit and Kebenesh Blayneh). The overall goal of my work during this period was to investigate the extent to which basic principles of classical theoretical ecological dynamics holdup (or do not hold up) when demographic structure within a population is included in the models— principles such as logistic equilibration, predator-prey oscillations, competitive exclusion, and so on. My growing involvement with population and ecological dynamics was greatly enhanced by the foundation of two programs in 1976 at the University of Arizona: the Interdisciplinary Program in Applied Mathematics (of which I have been a member since its inception) and the Department of Ecology and Evolutionary Biology (EEB), one of the first departments of its kind in the world. EEB had faculty members who were prominent advocates of mathematical modeling,

Figure 30. The Beatle Team at a meeting in Tucson, Arizona, USA, in 1998. From left to right: Robert Desharnais, Robert Costantino, Brian Dennis, me, and Shandelle Henson.

including Michael Rosenzweig (of predator-prey "paradox of enrichment" fame) and William Schaffer, both students of Robert MacArthur at Princeton University who is considered a founding father of ecology and evolutionary biology. Bill Schaffer made scientific headlines with his research in epidemics and especially from his claim that certain measle outbreak data displayed chaotic dynamics. Bill and I never published together, but we team taught graduate and undergraduate level modeling courses in the EBB and mathematics Departments and, as a result, I learned a great deal from him. That interaction sparked my first interest in chaos theory (along with the famous seminal papers of Robert May).

In 1989 and 1990 I attended and spoke at workshops and seminars at the Department of Environmental Studies, University of California at Davis, where I met biologist Robert Costantino, who was visiting there from the University of Rhode Island (to collaborate with Alan Hastings). Bob is an expert on flour beetles (*Tribolium sp.*). Besides being an agricultural pest, these beetles have been a laboratory experimental animal for over a half century. Bob and his former post-doc Robert Desharnais had done some modeling on (indeed wrote a book on) modeling the dynamics of *Tribolium* and approached me about a possible collaboration. The result was the creation of a team of four interdisciplinary researchers (the Beatle Team) consisting of Bob Costantino, Bob Desharnais and me together Brian Dennis, an ecologist and statistician. We we later joined by Shandelle Henson and Aaron King (see Figure 30).

A basic issue for the Beetle Team (yes, we were cognizant of the fab four ... the famous Beatles) was an uneasiness with the state of mathematical population ecology in that it was long on theoretical concepts and principles (stability, equilibrium, limit cycles, chaos, etc.) but short in convincing and predictive explanations of real biological systems. Ecologists rightly asked whether a low-dimensional, nonlinear (or even linear) population model had ever been a convincing representation of a real ecological system. We were hard pressed to find any examples in which population models constituted reliable explanations and predictions of population dynamic phenomena. This view was held by many, if not most, ecologists [3].

In 1991 we approached the National Science Foundation (NSF) with a grant proposal to establish a rigorous and descriptive, but more importantly, predictive model for the dynamics of a biological population (namely, the flour beetle), which would be validated by carefully designed, controlled, and replicated laboratory cultures. The main goal, in our mind, was to show that mathematical models, even low dimensional models, could be predictively accurate when confronted with data. NSF responded with a year's support to re-write the proposal so as to highlight the use of the model (if we could establish one) to demonstrate model predicted complex (nonequilibrium) dynamics and, in particular, a bifurcation route-to-chaos (a hot topic at the time). This we did and gained the support of NSF for over a decade, during which we developed and parameterized a discrete time (Leslie) model using historical data, studied the predictions of the model using analysis and numerics, determined bifurcation sequences that included chaotic dynamics and experimentally feasible manipulations that place replicated cultures along that route-to-chaos, and statistically validated (without re-parameterization) the model predictions against the observation data obtained. That project (which involved eight years of uninterrupted counting of hundreds of individuals every two weeks!) is considered the first unequivocal demonstration that a biological population can exhibit chaotic dynamics, as predicted by a mathematical model.

What type of mathematical model did the Beetle Team use? We began simple and made the model more complicated only when necessary to obtain an adequate statistical validation of model predictions (using already available historical data). The well-known Ricker model (which has the famous period doubling route-to-chaos) failed miserably to parameterize adequately (using maximum likelihood and conditioned least squares methods) for the dynamics of the flour beetle. We extended the model to include juvenile stages, as a time delay, but this two dimensional Leslie model also failed to parameterize adequately. A successful parameterization was obtained by including two juvenile stages, larval and pupal, and an adult stage (which our data luckily included). The result was a three dimensional Leslie model, now known as the LPA model,

$$\begin{cases} L_{t+1} = bA_t \exp\left(-c_{el}L_t - c_{ea}A_t\right) \\ P_{t+1} = \left(1 - u_l\right) L_t \\ A_{t+1} = P_t \exp\left(-c_{pa}A_t\right) + \left(1 - \mu_a\right) A_t \end{cases}$$

which served as a basis for the team's research for two decades. A report on the route-to-chaos project is given in the book [52]; also see [51].

The beetle chaos project accomplished much more than just a tight model-data fit and a demonstration of biological chaos. Numerous other dynamic phenomena were explained by the model, often with jaw-dropping accuracy (see Figure 31). Some of these model predicted phenomena were unknown to us prior to the experiments and some even seemed impossible or highly unlikely to the biologists, and yet were

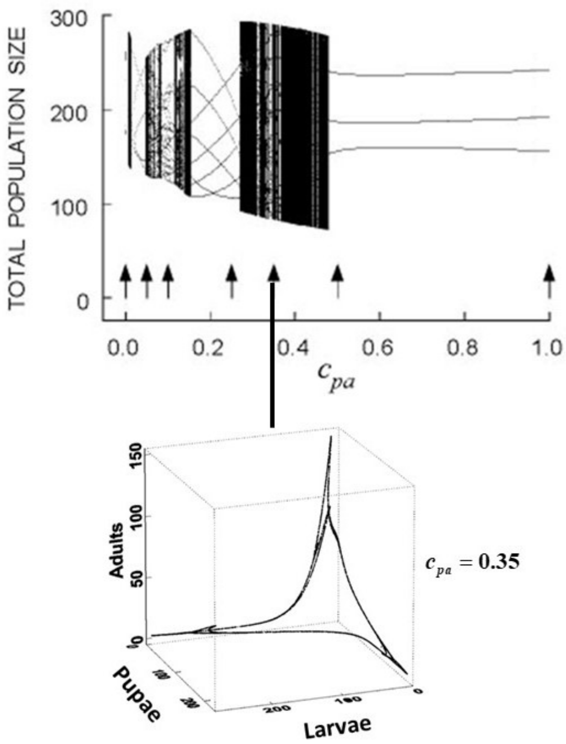


Figure 31. The bifurcation diagram from the LPA model that was used in the route-to-chaos experiments, conducted over a period of 8 years, shows numerous types of attractors, including a chaotic attractor at $c_{pa} = 0.35$.

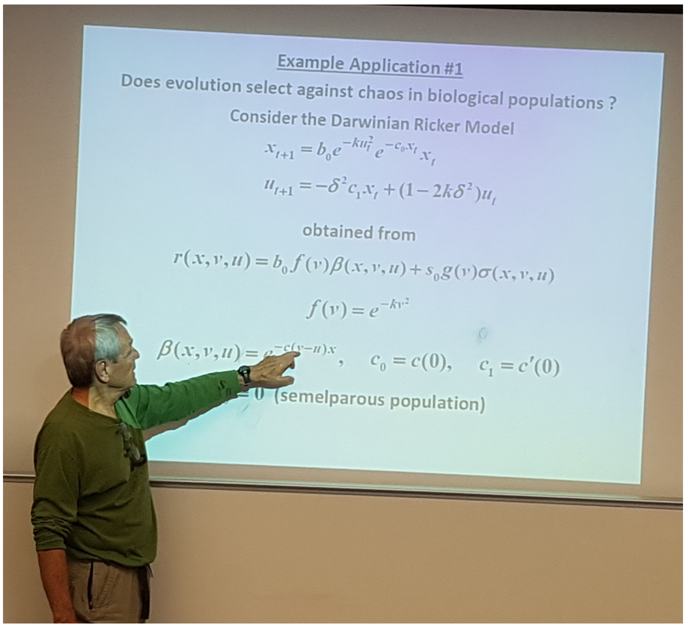


Figure 32. Lecturing at the 25th ICDEA conference held at University College London in 2019 on a Darwinian version of the famous Ricker model.

validated by experiments. The model explanation of many observations utilized variations of the LPA model: stochastic, periodically forced, coupled competing species, and evolutionary (Darwinian) versions (see Figure 32).

For example, repetitious patterns observed by Bob Costantino in the long data time series for the chaotic attractor corresponded beautifully to the unstable periodic orbits lying on the attractor [98]. The data followed orbits that (repeatedly, due to stochastic perturbations) fly-by the unstable (saddle) cycles on the chaotic attractor, a remarkable observation attesting to the accuracy of the *LPA model*. There was, however, a notable exception, namely, a distinct period 6-cycle pattern that is nowhere to be found on the chaotic attractor. An explanation for this cycle was found by realizing that all life cycle stages in real beetles come in whole numbers, but that the LPA state space is continuous. When the LPA model was placed on an integer lattice, we found that the attractor was then 6-cycle, in exactly the three dimensional phase space configuration observed by Bob. However, chaos cannot be present in a dynamical system on a finite state space lattice, since all bounded (deterministic) orbits are necessarily periodic. So, in what sense did the flour beetle experiment demonstrate the existence of chaotic dynamics? As shown in [87] it is a *stochastic version of the lattice LPA model* that explains the patterns observed in the experiment: although model predicted orbits tend to a 6-cycle, continual stochastic perturbations cause transient dynamics which, even though on a discrete lattice, have properties of the chaotic attractor of the underlying continuous state space LPA model (including fly-bys of the cycles on the chaotic attractor).

The stochastic lattice version of the LPA model is but one example of extensions of the model that provide remarkable explanations of patterns observed in the Beetle Team's experiments (see Figure 33).

Other examples include saddle fly-by's in non-chaotic dynamics and stochastic

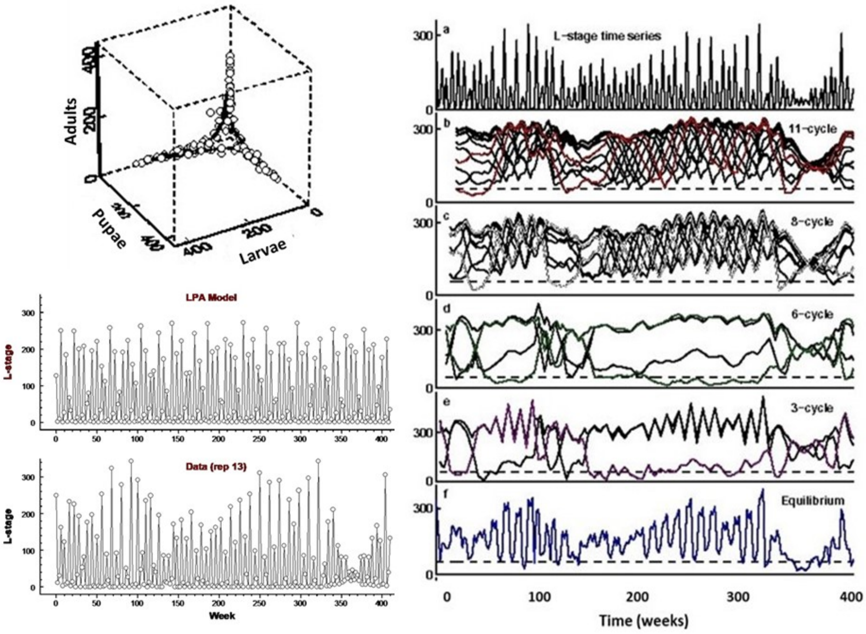


Figure 33. Beetle data, from one of the replicates set at the model predicted chaotic attractor, cluster around the attractor in state space. The time series of the larval classes from the LPA and the replicate data are compared, the latter of which shows a fly-by of the unstable equilibrium during weeks 350-400. The graphs on the right are lag metric measures demonstrating the beetle population flying by various unstable periodic orbits on the chaotic attractor of both the LPA model and the lattice LPA model.

phase shifts in oscillatory dynamics for stochastic versions (e.g. see [53]); resonance (as observed in experiments with flour beetles) and attenuance for periodically forced versions (e.g. see [46]); the evolution of genetic polymorphisms for Darwinian (evolutionary game theoretic) versions (see [154]); and challenges to the classical competitive exclusion principle based on coupled LPA models (see [56]). Much of this research is covered in the book [52]. A book that more comprehensively covers the work of the Beetle Team is currently in preparation (with Shandelle Henson as lead author); also see [55]. The LPA model and its numerous versions that we developed provide, to this day, many open and challenging problems for mathematicians interested in discrete time dynamics described by difference equations.

The research of the Beetle Team involved laboratory cultures of flour beetles. A rigorous connection between mathematical models and ecological data is obviously more difficult for populations in nature, where biological and environmental interactions are more abundant, complex, and not so controllable (if at all). My most recent interdisciplinary collaboration have been with Shandelle Henson and her husband, James Hayward, a field ecologist who has worked and gathered an immense amount of data on marine animals and birds on Protection Island, a United States Natural Wildlife Refuge managed by the US fish & Wildlife Service (see Figure 34). I was privileged to joint Shandelle and Jim, and their many colleagues in the Seabird Ecology Team, for over a decade of collaborations that modelled marine animal behaviour and population dynamics. While we use differential equation models for some projects, we also make use of difference equation models, particularly matrix models for structured populations and periodically forced and evolutionary game theoretic versions. Some

Figure 34. Members of the Seabird Ecology Team, hiking on the north shore of Protection Island, from left to right: me, Shandelle Henson, and James Hayward.

projects are in the spirit of the Beetle Team research in that, after obtaining from data statistical approximation of parameters and correlating them with environmental factors, we made predictions for what behaviour activities will be observed the following next season, predictions that were subsequently observed in the field. This work was done with nesting colonies of the glaucous-winged gull (*Larus glaucescens*). Other projects involve using models, in a proof-of-concept manner, to test hypotheses formulated from field observations. Of particular interest are phenomena related to climate change (specifically, rising mean sea temperatures in the Strait Juan de Fuca where Protection Island is located). For example, egg cannibalism by adult glaucous-winged gulls significantly increases during el Niño years (a kind of natural experiment for future climate warming). Such behaviour, it would seem, should be eliminated by natural selection over time; however, our models show that cannibalism can be (under certain circumstances) a long term, evolutionarily stable strategy. This result utilizes Darwinian versions of a discrete time matrix model for juvenile-adult (victim-cannibal) structured population. We have recently published a book covering the nearly two decades of interdisciplinary research by the Seabird Ecology Team [54].

My collaboration with the Beetle and Seabird Ecology Teams not only greatly affected my research career, but also my mentoring activities, especially with regard to discrete time matrix model analysis and applications in population, ecological, disease, and evolutionary dynamics. Postdoctoral students with whom I have worked, until my recent retirement, include Shandelle Henson, Olav Skarpaas, Aaron, King, Kehind Salau and Alex Farrell. Graduate students whose dissertations I directed include Martin Gildardo Garcia-Alvardo, Maref Alzoubi, Jeffrey Edmunds, Nakul Chitnis, Sheree Arpin (LeVarge), Rosalyn Rael, Suzanne Robertson, Amy Veprauskas, and Emily Meissen.

My road to dynamics has been a long one. And one that was quite varied, starting from early high school days of Euclidean dynamics and calculus, through pure mathematics training as an undergraduate, to applied mathematics and fluid dynamics in graduate school, and on to population, ecological and evolutionary dynamics with interdisciplinary teams of colleagues. After serving 53 years as a professor of mathematics at the University of Arizona, I am now retired with emeritus status. As a final word, I might add that during my travels on this long career path, I never stopped playing the piano. (I even worked with string students and their professors in the University's music department for several years.) And that I plan to keep doing so, as well as remaining active in research projects for as many years as I have left. The book [139] is published in 2023 for the 90-th anniversary(!) of the school.

9. My scientific youth with one-dimensional dynamics by Lyudmila Efremova

I am originally from Nizhny Novgorod (between 1932 and 1990, the town was named Gorky, in honour of the famous Russian writer Maxim Gorky, who was born there in 1868). In the mathematical (and (radio)physical) world Nizhny Novgorod is well-known for Andronov's school of nonlinear oscillations and dynamical systems. In 2023, for the 90-th anniversary(!) of the school, a recollection book [139] was published (see Figure 35).

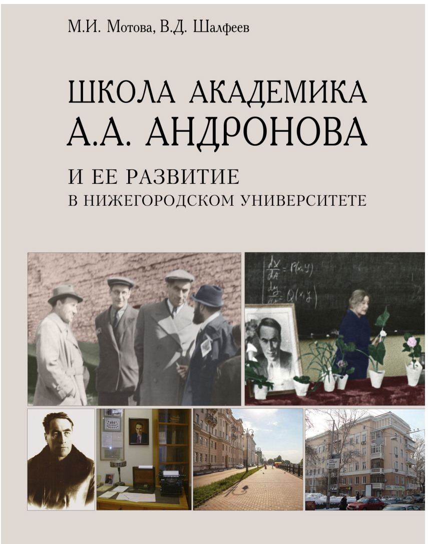


Figure 35. In the upper photo on the left, Academician of the USSR Academy of Sciences Andronov (second from right) and Professor Mayer (first from right) with Ph.D.-students (Ya. Nikolaev and S.Bellustin). In the upper photo on the right, Professor Leontovich-Andronova presents a report on Andronov's mathematical works.

Andronov's great merit is the use and development of fruitful, but forgotten (in the 20s and the 30s of the 20th century) Poincaré's ideas in the theory of differential equations for the needs of radiophysics and the theory of radio transmitters (see *e.g.* [18]). One of Andronov's ideas is based on the use the Poincaré secant (when it exists) for the description of trajectories dynamical behaviour of ordinary differential equations. Further development of this idea led to the creation of the "point mapping" method, which was integrated later into the theory of discrete dynamical systems. Here I will refer only to the articles on one-dimensional dynamics by Maier (he worked at Gorky State University) on rough (structurally stable) diffeomorphisms of a circle [127] and by N. N. Leonov (he worked at the Research Institute of Physics and Technology at Gorky University) on discontinuous maps of a straight line [108, 109].

It must be said that throughout its more than a century of history, Nizhny Novgorod (Gorky) University remains one of the largest educational and scientific centers of the country. On March 20, 1956, by a decree of the Presidium of the Supreme Soviet of the Russian Federation, Gorky University was named after the great Russian mathematician N.I. Lobaschevskii, native of Nizhny Novgorod.

When I was in school, I not only loved solving difficult mathematical problems, but also made attempts to write music (I also studied at a music school). One of my compositions for the symphony orchestra was performed at a major concert at my music school (to the delight of my parents). As a result, choosing which path to follow was very difficult for me. Nevertheless, in the struggle between my two serious hobbies, mathematics and music, mathematics won, and I entered the Faculty of Mechanics and Mathematics of the University.

When I was a 2nd year student, Dr. Rakhmankulov invited me to read the Sharkovsky's paper [187]. My impression was huge. I thought: "It is necessary to only know the classical Bolzano–Cauchy intermediate value theorem for continuous functions and nothing else! But how did he come up with his own "dynamical" order on the set of natural numbers? It is very beautiful!" Now after many years I can say that my naive first impression of Sharkovsky's theorem determined my scientific biography. Throughout my work in mathematics, from time to time I return to various generalizations of this wonderful theorem [60, 61]. In the third year, I received from Dr. Rakhmankulov the topic of my student research work. The exact setting of the problem was as follows. *Let f be a continuous circle map with a periodic point with period $q \geq 1$ and suppose that the map f has periodic points with period p for $p \neq q$. It is necessary to describe the set of all the (least) periods of f-periodic points.* Rakhmankulov himself solved this problem for $q = 1$ [155, 156], and I had to consider the cases of $q > 1$. At the time, attitudes to this problem oscillated between two extreme points of view: some believed that there was nothing to do here, since Sharkovsky did everything, while others said that it was impossible to solve this problem because it was very complex. I was young, carefree, and instead of these conversations, I listened with a feeling of happiness to lectures by major mathematicians (S.P. Novikov, Ya. G. Sinai) and physicists (V.L. Ginzburg, A.V. Gaponov-Grekov) who worked in Gorky or came here to give talks, as well as lectures by famous musicologists at the Gorky Conservatory on music theory, V.G. Blinova and M.M. Valentinov, the main director of the Gorky Opera and Ballet Theater.

My years as a student were filled with a constant sense of joy at learning new things. At the end of my studies, I obtained some partial results on the coexistence of periods of periodic points of continuous circle maps and graduated from Gorky University with honors. In graduate (Ph.D.) school, I continued to deal with the

Figure 36. The Department of Differential Equations and Mathematical Analysis of Lobaschevsky University (1975-76). In the front row, Rakhmankulov is first on right, Otrokov is fourth on right. In the second row, Efremova is fifth on left.

problem of the coexistence of periods of periodic points of continuous circle maps. Professor Otrokov and his former graduate student, Dr. Rakhmankulov, were my scientific supervisors (see Figure 36). Otrokov is a major specialist in the theory of limit cycles of differential equations. He was never a graduate student of Professor Andronov, although he always considered him to be his teacher. In 1940 Otrokov defended his Ph.D. thesis under Andronov's supervision. I assume that the influence of another famous mathematician, Professor Braitsev, specialist in the theory of functions, defined the fact that Otrokov used a functional approach for studying limit cycles. Otrokov had a vivid imaginative mindset, and communication with him always gave the prospect of further research.

When I was a Ph.D.-student in the 2nd year, Otrokov decided that it would be useful to show the results obtained to Professor Sharkovsky. So, after a phone conversation with Otrokov, Sharkovsky invited me to present the report at his seminar at the Institute of Mathematics of the Academy of Sciences of Ukraine. That was in 1977. Of course, I do not remember all the details of this report, but I remember my surprise that Sharkovsky was sitting at the back of the room where the seminar was held, not at the front, as seminars leaders usually do. I now know for sure that the back of the room is the best place to listen to reports! There was a lively discussion with Sharkovsky after that seminar, and I have had the opportunity to sometimes go to Kiev to present and discuss new results. It was always been very interesting!

It was at this time that the idea arose to apply the concept of the degree of mapping to the description of the periods of periodic points of continuous circle mappings [57, 58] (see also [39]). It was a fruitful idea which transformed the original setting of the problem, preserving it only for circle maps of degree 1. It turned out, for example, that the periods of periodic points of continuous maps of a circle of degree -1 satisfy the Sharkovsky's order (see the earlier works [39, 58]). As for continuous maps of a circle of degree 1, although the rotation of the trajectories of such maps was apparent in my proofs and examples at the time, I did not have enough experience to introduce the concept of a set of rotation for a circle map of degree 1. This was done later in the article [141]. Note that in [137] and [8] rotation numbers of individual periodic orbits are used.

Analysis of the examples I have built has led me to the idea of using a horseshoe when considering circle maps of degree 1 with periodic orbits of periods $q \geq 2$ and $p > q$ for $p/q \notin \{2^i\}, i > 0$. It was the second fruitful idea, which gave a form of representation of periodic orbits periods of circle maps of degree 1. This form looks like the decomposition of a vector by basis vectors [58, 62]; in addition, it turned out that the periods of periodic points of maps of degree 1 for $q = 1$ and $p > 1$ admit such a representation [63]. The use of the horseshoe also made it possible to prove the criteria for the existence of homoclinic points of circle maps of all degrees and analyze the role of transversality-type conditions for the existence of homoclinic trajectories of circle maps of degree 1 [58, 62]. Sharkovsky recommended this work to the Programme Committee of the Ninth International Conference on Nonlinear Oscillations (ICNO - IX, Kiev, 1981), and as a young scientist, I delivered a report on the results described above at this conference. It was the first major conference in my life, and I was very worried. Professor Plykin was the chairman of the section where I presented my report. Thanks to his goodwill, the report was successful. After that there were many interesting discussions, first of all with Professors Plykin and Szlenk. So, Szlenk informed me that the idea of representing the periods of periodic points of circle maps of degree 1 having periodic points with mutually simple periods $p > q > 1$, similar to the one described above, was also expressed by Professor Nitecki (unpublished).

As for Plykin, then later, in the collection of poems by poets of the science city of Obninsk, I saw his poems. So I found out that the author of the Plykin's attractor was also a poet! Moreover, in his young years Plykin was engaged in mountaineering and some mathematicians (for example, Professor Sinai) climbed mountains with him. A few months after ICNO - IX, I submitted and defended my Ph.D. thesis (see Figure 37) of which Sharkovsky and Belykh were the official opponents (referees). With great warmth, I also recall the very lively discussions of my Ph.D. thesis with Professor Belykh. Many years later, Belykh came out as one of the referees of my Doctor of Sciences thesis (physics and mathematics). Later, on the recommendation of Professors Plykin, Szlenk and Nitecki, I made a report on the above results at the Sinai seminar. The room in the main building of the Moscow State University, where the seminar was held, was crowded with people. Among the listeners were not only mathematicians and physicists, but also, as Plykin told me, meteorologists from the Hydrometeorological Center. As a member of the editorial board of the journal *Russian Mathematical Surveys*, Sinai submitted my short article to this journal.

While still in graduate school, I started working at the Gorky Pedagogical Institute. This work continued after defending my Ph.D. thesis in 1981 until 1987, when I received an invitation to work at Lobachevsky University at the department of Otrokov. So I went back to my home university and I am working there now.

After defending my Ph.D. thesis, I thought about continuing my work in the theory of dynamical systems. It should be said here that the last 25 years of the XX-th century were a golden period of one-dimensional dynamics. All the major achievements in this field relate to this period of time. But another period was coming, in

Figure 37. This photo was taken after the presentation of the results of my Ph.D. thesis at a seminar on differential equations led by Professors G.A.Leonov and Matveev at the Leningrad (St. Petersburg) Pedagogical Institute (1981).

which the most interesting areas of activity seemed to me, firstly, the applications of one-dimensional dynamics to the study of discrete dynamical systems on manifolds of dimension at least two; and, secondly, the creation of dynamical systems theory on complicated one-dimensional ramified continua, which do not allow order topology. I started studying skew products of one-dimensional maps on finite-dimensional cells, cylinders and tori, choosing the first line of research. The second problem was formulated for my first graduate student, E. Makhrova who is currently docent at Nizhny Novgorod University. She found very interesting and successful results in this direction.

At about the same time (in the early 90s of the XX-th century), representatives of Ukrainian, Czech, Spanish, Italian and other mathematical schools began to study the topological dynamics of skew products of interval maps. All these studies were aimed at solving the following problem. *Which properties of one-dimensional maps do skew products preserve, and which ones do not?* This approach allowed to construct examples of skew products of maps of an interval in dimension two with properties other than those of continuous interval maps. And although I also built examples, I was more interested in the question of why this is happening.

One of the main principles of A.A. Andronov was the principle of “mobilization of information” [139]. Therefore, when I started working on skew products of maps of an interval in dimension two, I read, in particular, all the works on the study of cylindrical cascades, starting with a small text in Poincaré’s memoir [149]. And again, among the ideas of Poincaré (he is truly inexhaustible!) there was one, the modification of which, in relation to the skew products of the maps of the interval in dimension two, made it possible to conduct a systematic study of such maps [59]. Professor Anosov, Academician of the Russian Academy of Sciences, was the first mathematician who drew attention to the relationship between the study of cylindrical cascades and the solution of Hilbert’s 5-th problem. In the process of studying skew products of interval

Figure 38. At the conference "Mathematical Physics, Dynamical systems and Infinite-Dimensional Analysis - 2023" (Dolgoprudny, Moscow Region, 2023). Efremova is forth from the left, Makhrova is third from the left.

maps, it also turned out that the study of such maps adjoins the range of issues related to Hilbert's 13-th problem.

In 2003, the book "Mathematical Events of the XX-th Century" was published in Russian, under the editorship of Academician of the Russian Academy of Sciences, Professor Arnold. This book opens with an article by Anosov, "Dynamical Systems in the 60s: The Hyperbolic Revolution" [15]. The article made a powerful impression on me. This is a mathematical article, in which the author managed to convey the experiences of a person creating a completely new knowledge!

It was at that time, when the idea arose of my internship at the Steklov Mathematical Institute of the Russian Academy of Sciences under the supervision of Anosov. The opportunity for a two-week internship was presented only in 2009. Anosov agreed to be the scientific supervisor of my internship. In that time the feeling of happiness of learning new things returned to me, which I experienced during my student and graduate years. There was another two-week scientific internship under the guidance of Anosov in 2011. And all this time since 2009 I interacted with Anosov (until 2014, when he died). I have repeatedly made presentations at the seminar "Dynamical Systems and Ergodic Theory" under the guidance of Anosov and Stepin, where the results were thoroughly discussed.

I want to tell only one story here, which vividly characterizes the high human qualities of both Professors Anosov and Stepin. One day I called Dmitry Victorovich Anosov in the autumn of 2013. He was already feeling unwell, and during the conversation he suddenly asked me: "Which of the Moscow mathematicians would you like to cooperate with?" After some silence, I said that I would like to cooperate with Stepin. Then the conversation turned to another topic, and Anosov did not return to this issue anymore. And it was only from Stepin's speech at the defense of my Doctor of Sciences Thesis in 2018 that I learned that Anosov spoke to him and asked him to treat me carefully and help me. And this was done by a man who understood that he was leaving... Even now, when I write about it, it seems incredible to me! Stepin fulfilled everything that Anosov had talked to him about shortly before his death. In

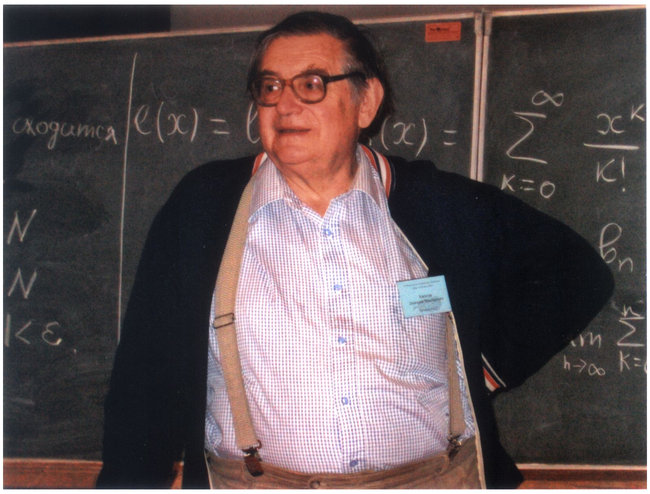


Figure 39. (This is a photo from the family archive of D.V. Anosov). D.V. Anosov gives a lecture at the Workshop "Contemporary Mathematics" in Dubna.

Figure 40. The photo was taken by Professor Ryzhikov before the meeting of the seminar on ergodic theory and dynamical systems. In the front row, Professor Oseledets is first on right, Professor Gurevich is first on left. In the second row, Professor Anosov is the first on right, Professor Stepin is first on left.

Figure 41. The photo was taken by Professor Ryzhikov. Professor Stepin is listening to a talk at the seminar "Ergodic Theory and Dynamical Systems".

particular, he submitted my survey on skew products of maps of an interval in dimension two to the editorial board of the journal "Russian Mathematical Surveys", and after a careful review the article was published. He then recommended this work to Ya. G. Sinai (Abel Prize, 2014). After that, I exposed my Doctor of Sciences thesis at the seminar of the Dobrushin Laboratory at the Institute of Information Transmission Problems of the Russian Academy of Sciences. The work was accepted, and I defended my Doctor of Sciences thesis in 2018 at this Institute in the Council, which was chaired by Sinai. The official referees of my Doctor of Sciences thesis were Belykh (one of the most prominent representatives of the school of Academician Andronov), Professor A. Zhirov (a former Ph.D. student of Professor Plykin) and the famous Russian topologist Professor S. Bogatyi.

In 2018 I received an invitation to work at the Moscow Institute of Physics and Technology. Currently, I am also full Professor at this Institute. This work, communication with specialists in mathematical physics attracted my attention to the study of quantum mechanics, and my husband Professor Sakbaev and I applied the technique of studying the Ω-blow up in skew products of interval maps to the classification of blow ups in the set of solutions of differential equations [64].

My active work in low-dimensional dynamics continues. Along the way, I met many outstanding people who played a big role in my scientific biography. I remember this with great warmth and gratitude. I am writing this article thinking about a new generation of mathematicians. They are different, but I get a great pleasure from working with the last generation of my students. When I see the sincere interest in their burning eyes, I believe that the development of the theory of dynamical systems and its applications will continue.

The author thanks Dr. E. Andronova, Dr. A. Klimenko, Dr. E. Makhrova and Professor V. Ryzhikov for their help in the selection of photographic materials.

10. How I got caught up in chaos studies by René Lozi

Is everyone's life predestined? this question has troubled humanity for millennia. Personally, I don't think so, but I wonder about the circumstances that can lead someone to choose their path in life. The term "math bump" originates in the 19th century, in a pseudo-science called "phrenology". Founded by the German neurologist

Figure 42. This photo is taken during the Conference on Dynamical Systems (Atlanta, USA, 2011). From right to left: Professors Fournier–Prunaret, Efremova, Sakbaev, my Ph.D. student Andrey Filchenkov.

Franz Joseph Gall, it argues that brain capacities are distributed in well-defined areas of the brain, and that the shape of the skull then reflects these capacities. According to this false theory, which is still more or less popular, people gifted in mathematics would have a visible "bump"; or at least this expression is used in a metaphorical sense. A friend of my mother's touched my baby's head, and she decided that this was the case for me. This preposterous statement was often repeated to me during my early childhood, and I wonder if hearing it convinced me that I had a predisposition to study mathematics!

What is remarkable is that, like Michał Misiurewicz did and does today (Section 2), I counted many things around me during my childhood and until now (how many friends I had at school?, how many seconds is a year?, a century?, how many stairs I have to climb to reach my office?, etc.). My whole family laughs about it! Besides, 70 years later, I also have bad memories of my preschool year: my teacher didn't let me play during recess. She made me do math exercises that the other students weren't able to do. When I was 13, one of my cousins, an engineer, introduced me to the base 10 logarithms. I was fascinated. I constructed a cardboard slide rule, grading it with my older sister's logarithms chart, and used it at school, much to the astonishment of my classmates and the teacher. Forty years later, many of them still remember it!

I feel sometimes uneasy to recall what exactly happened nearly fifty years ago in my academic life. In what follows, I hope to reconstitute fairly accurately the beginning of my research in bifurcation and chaos theories at University of Nice (now University Côte d'Azur) where I spent all my life, even under different administrations of research (C.N.R.S.) or education. In fact, my office was always inside or in the same small neighborhood of the Faculté des sciences, where stands the Institute for mathematics Jean-Alexandre Dieudonné, in honor to the famous "Bourbakist".

Being a mathematician is very special, not only do people think you are different from normal people, not keeping your feet on the ground; but you can develop a decades-long friendship with researchers from other countries, speaking the same common mathematical language. This is why, beyond mathematical considerations, I will take the time to talk about friends of mine and outstanding mathematicians that I have met over the years.

I started my studies at the University of Nice – which is a distance of 20 km from Antibes, my home town – in October 1967, in mathematics and physics. It was classical studies in a world that had not known many upheavals since the second world war, except the conquest of space started in 1957 with the launch of the Russian satellite "Sputnik" and the journey of the first astronauts, the Russian Yuri Gagarin in 1961 and the American John Glenn in 1962. I was fascinated by particle physics and I dreamed to make researches in theoretical physics, without knowing which courses to attend to reach this goal. At the university I took my first programming course about FORTRAN IV in 1968, using punched cards. The decades that followed and of which I will try to describe the evolution that I personally perceived in the limited field of the study of bifurcations and chaos, has seen great changes, such as the conquest of space with Neil Armstrong, the first man who walked on the moon on 20 July 1969, the wide spread of Personal Computers, wireless phones, the birth of Internet, Artificial Intelligence, the easy plane travels, etc.

During my studies at the University of Nice, I took at the same time courses in theoretical physics and applied mathematics. Graduated with a certificate in fields and particles, I no longer liked this discipline but I discovered with fascination the methods of numerical integration of ordinary differential equations (ODEs). It was during the academic year 1970-71, I was following my bachelor's degree under the supervision of Professor Martin Zerner (1932-2017). During my previous studies, I had been taught that there was a list of ODEs written by Bernoulli, Lagrange, Clairaut, Riccati, etc. and a list of solving methods. A clever student in mathematics should be able to know which method and which trick was efficient to solve a given equation among a huge list of them available in handbooks. No physical sense, in fact no meaning at all, was attached to these academic exercises. No numerical method was taught. Moreover, between professors, there was a strict separation between "pure mathematicians" and the few "applied mathematicians" who were able to use a computer. It must be said that only one computer "IBM-1130" was available at the university of Nice (another computer IBM-7040 was installed at the Observatory [5]). Martin was considered as weird by the other professors because he was using a computer (although his Ph.D. superviser was Laurent Schwarz and he was chosen, when working as "attaché de recherches" at C.N.R.S., to help Jacques Hadamard to finalize the last books of his complete works in 1958-59). I knew him fairly well, because during the famous "student revolution" of May 1968, students and professors of the small faculty of science of Nice had exchanged a lot, breaking down the usual hierarchical barriers. With Martin we even guarded together, against hypothetical attacks, the "Château de Valrose", building of the Institute of mathematics, during many nights of this month of May, perched on its roof, like medieval warriors in their dungeon.

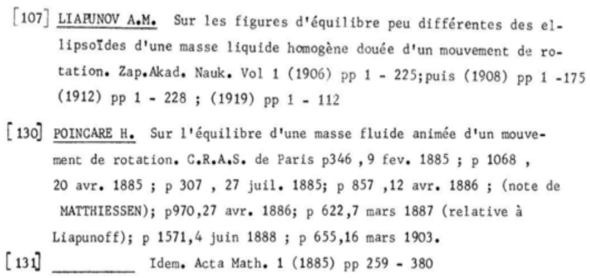
[5] I used it for my Ph.D. facing the same problems encountered by Eckehard Schöll (Section 5.)

During his lectures, my mind knew a breakthrough that changed the paradigm in which I conceived the theory of differential equations: the set of all the equations I was taught, were of zero measure in the set of all ones existing. No closed formula of solution can be found for most of them. Only numerical methods like Runge–Kutta, Adams-Bashforth, etc. were able to provide approximate solution. Of course, in this scope, computers were essential. Moreover, ODEs were useful to model physical, chemical or even biological situations (at the end of the year, to be graduated, I presented a memoir on the Yukawa potential [202]). This new paradigm has guided my research career throughout my whole life. Looking for a supervisor for my Ph.D. thesis in 1972, I had the chance to meet Professor Giuseppe "Pippo" Geymonat, professor at the Politecnico in Turin (Italy) in sabbatical in Nice, who first supervised my "Diplôme d'Études Approfondies" and proposed me to study the particular nonlinear ODE

$$\begin{cases} u''(x) - \left[u'(x)\right]^2 + cu'(x) = 0, \\ u(0) = u(1) = 0 \end{cases} \tag{8}$$

which belongs to the class of two-point boundary value problems [96].

The study of (8) led me to defend in 1975, my Ph.D. thesis, formally supervised by Jean Céa, entitled "Analyse numérique de certains problèmes de bifurcation" [115]. Just after the defense I published a short article on, perhaps the first example of continuation method, at least in the context of the bifurcation theory [116]. While preparing my Ph.D., the name "bifurcation" was largely unknown in the communities of mathematical and numerical analysis in France. When people in Nice and other universities were asking me on my research topics, they were astonished by this name. Of course, as one can read in Abraham's [110] that the term bifurcation was introduced 90 years before, by Henri Poincaré [150] who used the term "form of bifurcation" to describe a form of equilibrium belonging to two different linear series of differential equation, which does not correspond exactly to what it is referred nowadays as the solution to nonlinear differential equation. Furthermore, we must consider that the decade 1960-70 was in France the golden age of the Bourbaki group, whose philosophy was drastically opposed to Poincaré's way of thinking. The bedside books of every young student in maths at university were "Éléments de mathématiques" [41]. Moreover, Jean Alexandre Dieudonné, one of the founders of the Bourbaki group, arrived at Nice in 1964, just after the University of Nice was created. He was the most prominent professor from the department of mathematics. During the academic year 1969-70, I attended his lectures about the integration theory, with only a dozen of students. Three years after, upon Martin Zerner's request, he spent an entire afternoon in helping me on some problem related to my doctoral work. Poincaré's works were therefore not at all in my mind, although I eventually quoted him [150] in the analytical bibliography of my Ph.D. thesis (Refs. [130, 131] in [115]), together with the papers of A. M. Lyapunov (Ref. [107] in [115]) (see Figure 43). Abraham [110] referring to [197] mentions also that "The word bifurcation is indexed by Thom on 18 pages," but René Thom, Fields medalist in 1958, was doing, in my opinion, theoretic mathematics at a level I never could reach. Moreover, his book in English was published during the middle of my thesis, and at this epoch, it took months, or rather years, for a new published book in English will be delivered to our library. Oddly, few years later beginning in 1982, Thom assisted me many times for my "Thèse d'État," later for my Habilitation and



[107] LIAPUNOV A.M. Sur les figures d'équilibre peu différentes des ellipsoïdes d'une masse liquide homogène douée d'un mouvement de rotation. Zap.Akad. Nauk. Vol 1 (1906) pp 1 - 225;puis (1908) pp 1 -175 (1912) pp 1 - 228 ; (1919) pp 1 - 112

[130] POINCARE H. Sur l'équilibre d'une masse fluide animée d'un mouvement de rotation. C.R.A.S. de Paris p346 ,9 fev. 1885 ; p 1068 , 20 avr. 1885 ; p 307 , 27 juil. 1885; p 857 ,12 avr. 1886 ; (note de MATTHIESSEN); p970,27 avr. 1886; p 622,7 mars 1887 (relative à Liapunoff); p 1571,4 juin 1888 ; p 655,16 mars 1903.

[131] __________ Idem. Acta Math. 1 (1885) pp 259 - 380

Figure 43. The references to Lyapunov and Poincaré in the bibliography of my Ph.D. thesis.

to apply for higher position in the university [6].

Hence, the first paper in French on this subject (however in the fields of mechanics) was published in 1971 by Gérard Iooss [88] who obtained later, in 1974, a permanent position in the University of Nice. After my Ph.D., I asked him for a new subject for my "Thèse d'État." We hence worked together on the famous dynamo problem explaining the origin of the magnetic earth field [90]. During this period, we both attended the "International Conference on Mathematical Problems in Theoretical Physics," organized by the university of Roma, Italy (June 6-15, 1977), with a team of researchers from the university of Nice (Claude Bardos, Alain Chenciner, Pierre Coulet, etc.). The opening talk of this conference was given by David Ruelle [159], famous for coining the name "strange attractor" with his colleague Floris Takens [158]. In his talk, Ruelle conjectured that, for the Hénon attractor, the theoretical entropy should be equal to the characteristic exponent. It is how I learned the first example of chaotic and strange attractor (6) whose fractal properties were highlighted by Hénon and astonished the research community. Although Michel Hénon was astronomer at the Nice Observatory, few kilometers abroad from my office, I did not know him at this time.

Beyond bifurcation problem, my main interest was focused to discretization problem and finite element method (f.e.m.) in which nonlinear functions are approximated by piecewise linear ones. During the Roma conference, I tried unsuccessfully to apply the spirit of the f.e.m. to the Hénon attractor. Back to Nice on June 15 in the morning, during the defense of a Ph.D. by a colleague of mine, I eventually decided, using paper and pencil, to change the square function of the Hénon attractor, which is U shaped, into the absolute value function, which has a V shape, implying folding property. Few minutes after, in my office, I tested this modification, on my small desktop computer HP 9820 linked to the HP 9862 plotter. I shifted the parameter value a from 1.4 to 1.7 and b from 0.3 to 0.5 (why? I do not remember!) and plotted what is known today as the "Lozi map". Iooss and Chenciner encouraged me later to publish the formula (7) which appeared in the proceedings of a "Meeting between physicists and mathematicians about nonlinear problems and their applications" held in Nice on 26-30 September 1977 [117]. This was for me the very beginning of my career in chaotic dynamical systems. I was convinced that few weeks would be enough to explain and give a proof of the structure of a so simple attractor, but I failed.

[6] I'm impressed by what Smale wrote about Thom's influence at the beginning of his career, in 1956 [193].

In the next two years I attended two workshops on iteration theory: the first one on 21–23 May, 1979 at La Garde-Freinet (a small town in the south of France), where Alain Arnéodo, Igor Gumowski, Michel Hénon, Christian Mira and Yves Pomeau were also present, and where Michał Misiurewicz, after some questions at the end of my talk, jumped on the stage, and on the blackboard gave some clues of his forthcoming results presented at the famous New-York conference, seven months later [135]. The second meeting is a summer school in physics on July 1979 in Cargèse (Corsica) where I used the genuine non differentiable Lozi map to prove the existence of one homoclinic point for a smooth version of it

$$T_{z_\epsilon}(x,y) = (y + 1 - az_\epsilon(x), bx), \tag{9}$$

$$\begin{cases} |z_\epsilon(x) - |x|| \leq \epsilon \quad for\,|x| < \epsilon, \\ z_\epsilon(x) = |x| \quad for\,|x| \geq \epsilon, \\ z_\epsilon(\cdot) \in C^k, k \geq 2, \end{cases}$$

with ϵ small enough.

Then, applying a theorem proposed by Stephen Smale [192], I proved the existence of an invariant Cantor set for (9) [118]. Before these two workshops, I met Smale in Orsay, and at the end of his talk, I discussed with him about the validity of my proof. During the workshop in Cargèse, I sympathized with Joe Ford from Georgia Institute of Technology who, in the 70s, conceived the idea of disseminating *Nonlinear Science Abstracts*. Eventually his efforts resulted in the foundation in 1980 of the first journal devoted to nonlinear dynamics, *Physica D* [201]. I met him again in New-York (see below) and during the Dynamics Days, Twente University, May 25–26, 1982. After Cargèse, took place the International Conference on Nonlinear Dynamics, patronized by the New-York Academy of Sciences, on December 17-21, 1979, in the Barbizon–Plaza Hotel theatre, near the south bound of Central Park. During this conference I am proud to have shaken the hand of Edward Lorenz [7], the father of chaotic attractors, and I listen with a mix of anxiety and curiosity to the first proof by Misiurewicz for the existence of a chaotic attractor for the map I discovered two and half years before [135]. The speakers and the session chairmen were the worldwide foremost figures of nonlinear analysis or experimenters in physical sciences or chemistry: M. V. Berry, M. J. Feigenbaum, J. Ford, J. P. Gollub, J. Guckenheimer, D. D. Joseph, A. Katok, J. B. Keller, N. Kopell, O. Lanford III, P. D. Lax, E. N. Lorenz, B. B. Mandelbrot, J. E. Marsden, M. Misiurewicz, J. Moser, S. Newhouse, Y. Pomeau, M. I. Rabinovich, D. Ruelle, M. Shub, Ya. G. Sinaï, S. Smale, E. A. Spiegel, H. L. Swinney, Y. Ueda, C. Vidal, S. M. Ulam, J. Yorke, and many others. From Nice, Uriel Frisch, Claude Froeschlé, Michel Hénon (these three from the Observatory of Nice), Gérard Iooss, Charles Tresser, as I remember, attended the conference. I knew personally some speakers: Ford, Frisch, Joseph, Marsden, Misiurewicz, Moser, Newhouse, Pomeau, Ruelle, Smale, Spiegel, and Vidal. The topics of the conference was split in six sections: Turbulence, Ergodic and integrable behaviour, Physics and chemistry, Chaotic maps and flows, Chemical and fully developed turbulence, Strange attractors.

[7] 28 years after the New-York conference, Lorenz mentioned the Lozi map in his last article [114], publicated after his death.

I was interested in the session devoted to turbulence due to the concept of strange attractor developed by Ruelle and Takens [160] to overcome the classical theory of Landau promoting an infinite sequence of bifurcations in tori of increasing dimension corresponding to quasiperiodic motion with an increasing number of frequencies [103]. The talk by Vidal [204] on the Belousov-Zhabotinsky reaction was of a so great interest for me, because I had visited in the early 1979 his Paul Pascal Institute in Bordeaux, during which Claude Lobry who was teaching there, at the mathematics department, introduced me to the team of chemists working on this oscillating reaction.

Of course, the talk by Misiurewicz [135] was a kind of ecstasy for the young researcher that I was. Surprisingly the talk was entitled "The Lozi mapping has a hyperbolic structure," differently from the corresponding paper in the Annals, and nested in the session "chaotic maps and flows," not in the last session "strange attractors." Even if it was possible to listen every lecture, I did not attend all, nor did Michel Hénon that I met many times visiting computer shops in the streets of New-York city. One has to say that in 1979, there was a huge technological gap between France and USA. USA was a kind of *magic kingdom* for me. I saw for the first time microwave oven and cloth dryer, one year before in California. Therefore, Hénon who was a geek before the word "geek" was invented was looking for to buy a new pocket calculator, the HP 25 which was then unavailable in France.

Beside the interest of the scientific program, the most memorable memory of my stay in New York is to have crossed the city by night, after dinner, walking during three hours from Battery Park to Central Park, via Greenwich Village, with my colleague Claude Froeschlé! I remember also the first snowfall of the winter with 3.5" of snow fell, on Wednesday 19, on New-York City which forced me to buy warm clothes and gloves! [210].

Let me go back to the early 70s, and, also to fill the gap between July 1977 and May 1979. As I said above, in France, the name bifurcation was largely unknown in the math community of analysts at the beginning of this decade. Referring to [110], the situation was greatly different in the U.S.A. However, the situation was slowly changing in my country. At the end of March 1973, on the incentive of Geymonat, and with the help of Zerner, I invited to give a seminar in Nice, Paul H. Rabinowitz, a former student of Jürgen Moser (the M of the KAM theorem), who travelled from the "Scuola Normale di Pisa" to Paris. He spoke about his last results "Bifurcation from simple eigenvalues" [47]. However, the name bifurcation was not yet very popular, it was in competition with "nonlinear eigenvalue problems," "branching" and "buckling" (and "diramazione" in Italian).

On June 16-25, 1974, I attended a C.I.M.E. (Centro Internazionale Matematico Estivo, sponsorized by NATO) seminar in Varenna, Italy, entitled "Eigenvalues of nonlinear problems." The title of the talk given by Rabinowitz was "Variational methods for nonlinear eigenvalue problems" [153]. The lecture by R. E. L. Turner was entitled "Positive solutions of nonlinear eigenvalue problems" [199], Paul Fife, in his talk about "Branching phenomena in fluid dynamics and chemical reaction-diffusion theory" [70], indicated that "The study of the bifurcation and branching behaviour of problems in partial differential equations is a relatively new endeavor, and is rapidly gaining momentum."

The arrival of Iooss in our department of Mathematics in September 1974, was the turning point of the development of the bifurcation theory in Nice [89]. He was in touch with many researchers working on the topic including Klaus Kirchgässner, Daniel D. Joseph, Hans True, and many others. Alain Chenciner arrived the next year and stayed three years in Nice. Some researchers from U.S.A., were visiting our department, like J. F. K. Auchmuty [20] and H. B. Keller [95]. Nevertheless, the study of bifurcation phenomena was often marginally nested in workshop, seminar or conference of nonlinear analysis and application or physics. From 18 to 24 January 1976, I was invited by Kirchgässner with Iooss and Claude Bardos (visiting professor at Nice in 74–76), at Mathematisches Forschungsinstitut Oberwolfach for a seminar on "Nichtlineare Funktionalanalysis und irhe Anwendungen,"[8] organized by him, H. Amann (Bochum) and N. Bazley (Köln) [9]. I spoke about the numerical computation of branches of bifurcation [115] and Iooss about the "Conjectures on the dynamo problem for the geomagnetic field" [90]. Some bifurcation problems were exposed by N. Chafee (Hopf Bifurcation and Arbitrary Perturbations of a Differential Equation), E. N. Dancer (Bifurcation from infinity), R. J. Magnus (The odd multiplicity criterion of bifurcation theory) and R. Nussbaum (A Hopf Global Bifurcation Theorem for Retarded Functional Differential Equations). There were only six talks on bifurcation among a total of 27.

All the walls of the conference room at Oberwolfach support gigantic blackboards. An amazing time during this seminar was when, at the end of the talk by Bardos (Hölder estimates and time regularity for the Euler equation), all these blackboards were covered with formulas and theorems and that Ersnt Hölder (Mainz), the son of Otto Hölder, the mathematician whose name was given to the Hölder spaces, stood up, went to the blackboards and before each word "Hölder" wrote either E. or O., with reference to his father's work, or to his own! Unfortunately, I did not understand the explanations he gave in German.

When as young student, I studied analysis, the name of the mathematicians given to the functional spaces, like Frechet, Banach, Hilbert, Hölder, Sobolev, Besov, had a scent of history, except for Sobolev (1908-1989) who, Céa reminded me from time to time, attended to the first colloquium on numerical analysis he had organized in Paimpol (France) in 1967 [9]. However, concerning Hölder, I simply realized that he was not so far in the past as I imagined.

In September 1976, after two years as an assistant (the first grade of permanent position in the French university) I obtained one of the very few positions in France at C.N.R.S. as "Attaché de Recherche" with Céa for superviser. In September 1977, the group around Iooss, Chenciner, Coste, Frisch, Coulet, Tresser, was sufficiently strong to organize a "Meeting between physicists and mathematicians about non-linear problems and their applications," gathering around thirty speakers, includin Hénon [86]. In June 19-27, 1978, this group attended the C.I.M.E seminar on Dynamical

[8] Nonlinear functional analysis and its applications.

[9] Céa, who was very close to Jacques-Louis Lions, his former supervisor, was at that time professor at the university of Rennes. He said me recently that Lions phoned to him to invite Sergueï Lvovitch Sobolev, visiting professor at Collège de France, at this colloquium. Sobolev was fluent in French and knew a lot about the history of France. He was willing to meet young students. At first, in Paimpol, all the researchers wanted to stroll with him in the city, but soon they changed their mind because Sobolev was reading the names of the streets and told a lot about the incumbents, while the French mathematicians did not know so much. So, not to show their ignorance, there were fewer candidates to walk with this great man.

systems in Bressanone (Italia). Chenciner gave one of the five lectures during this event [42], along with the reputed researchers Guckenheimer, Moser, Misiurewicz and Newhouse. I sympathized with Newhouse and I drove him in my car to the nearby mountains of Trentino. On July 10-11 of the same year, Marsden [131] organized a "Symposium on nonlinear analysis and mathematical physics" at the university of California, Berkeley, in the Evans building recently achieved. I attended this workshop together with Chenciner, Bardos, Iooss and Jean-Michel Lemaire (geometrician) from Nice. I remember that David Chillingworth who studied also the dynamo reversal of the magnetic Earth field, was present [43]. It was my first visit to this university and my first stay in the USA. During a picnic lunch, Marsden tried to initiate us to play frisbee, however the French team was never able to play as a true team, like our American colleagues.

In October 1978, Céa, incited me, from an idea of J. L. Lions, to organize a conference on bifurcation in Nice. The Institute of mathematics provided all the funds. I managed a 4-days conference entitled "Didactique et recherche en bifurcation", on December 4-7. Among the participants was Jean-Christophe Yoccoz (Introduction to the work of Pesin on differentiable manifolds). It was the first ever invitation of Yoccoz, future Fields medalist (1994), to give a talk in a conference. In fact, I tried to invite his well-known superviser Michael Herman that I visited in Paris; however, Herman was not comfortable in travelling by train (at this time, travelling from Paris to Nice by air was very expensive). He said me "I know a very gifted student in third year of the École [Normale Supérieure (Paris)], he can speak as well as me." [10]

During the year 1979, I worked on the research of a realistic model for chaos, coming either from physics or chemistry. At this time, I encountered Claude Lobry, at the university of Bordeaux (he moved the next year to Nice), who was working with the group directed by Adolphe Pacault on the Belousov-Zhabotinsky reaction at the Paul Pascal Institute including Christian Vidal and Jean-Claude Roux. Then I collaborated for several years with him. We were interested in "slow-fast" mathematical models for the Belousov-Zhabotinsky reaction, like the Brusselator model made of two ODEs developed by Nicolis and Prigogine in 1977 [142], the Oregonator devised by Field and Noyes working at the University of Oregon which is composed of five coupled elementary chemical stoichiometries [69], and the model with even more ODEs developed at Bordeaux [203, 204]. To model the complex bifurcation diagram observed in very precise chemical experiments in Bordeaux, I introduced the idea of bifurcation of "motifs" (patterns) in a note to the Académie des Sciences de Paris, presented by Thom [119]. This was my first indirect exchange with Thom. I developed this model with Lobry [112], and on June 29, 1983, I defended my "Thèse d'État" that Thom accepted to review [121]. He was very interested by the ontology of the slow-fast metaphoric model I introduced.

I had also a long scientific relationship with José Argémi (1933-1985) [17] who worked at C.N.R.S. in Marseilles and his doctoral student, Bruno Rossetto (now professor emeritus) from the University of Toulon. Argémi was a specialist in slow-fast ODEs systems. He coined *co-folded line* and *pseudo-singular point*, very

[10] In his thesis, Yoccoz improved theorems of his supervisor Herman by giving simpler proofs but also obtaining the same results under weaker hypotheses. After that, he made major progress on an important conjecture about Benoît Mandelbrot's fractal ensemble, leading to a conjecture by A. Douady and J. Hubbard, which earned him the Fields Medal in 1994. In this aim, he introduced the famous "Yoccoz puzzles" [126].

useful mathematical tools for such systems. During 7-17 September 1982, the three of us attended the colloquium “Logos et théorie des Catastrophes” organized by Jean Petitot to celebrate René Thom in Cerisy–la–Salle [147]. Numerous renowned mathematicians and physicists like Michael Berry, Chenciner, Chillingworth, Adrien Douady, Georges Lochak, Ruelle, Timothy Poston, Bernard Teissier, and Sir Cristopher Zeeman gave a talk. All those, and scientists of other disciplines, philosophers, psychoanalysts, linguists, were very impressed by Thom. Me too much more. It was my second in-person meeting with him. The first one took place in May of the same year, during a meeting in Toulouse [120]. It was precisely during the colloquium in Cerisy–la–Salle, where I gave a talk entitled “Bifurcation de motifs dans la réaction de Belousov-Zhabotinsky”, that Thom accepted to review my “Thèse d’État”. During the speech of conclusion talk Thom said [147] (p. 513)

“ [...] So that even where there is apparently chaos, there are always morphological elements to study. The presentations that we have heard subsequently, for example that of Lozi very recently, are in exactly the same direction: there is always interest in trying to understand how things evolve morphologically and, if we cannot, of course, one can always dive into statistics.[11]”

Thom was very kind to suggest to Ruelle to invite me to give a talk on the same subject during a meeting on week turbulence held at I.H.E.S. on November 18-19, 1983. In May 1989, I was one of the height mathematicians (with M. Chaperon, A. Chenciner, J. Martinet, J.P. Ramis, P. Moussa, R. Moussu, F. Pham,) invited to pay tribute to Thom on his retirement during the “Rencontre entre mathématiciens et physiciens théoriciens” in Strasbourg [122, 157].

During spring 1981, I received the surprise visit of Hiroshi Fujii [72] from Kyoto university, who was spending one year in INRIA near Paris. He was interested by my Ph.D. dissertation. He gave me the names and addresses of Shigehiro Ushiki, and his superviser Masaya Yamaguti [207], both also from Kyoto university. I sent to Ushiki some papers on July 1981 who, in turn, sent me back his own ones on August 1981. I met him one year later, during the meeting in Toulouse. He was fluent in French. Then we worked very closely for twelve years, first on slow-fast systems, and hereafter on the Chua circuit.

In 1983, thanks to Professor Yamaguti, I got three weeks grant from the Japanese Society for the Promotion of Science for an internship at the Faculty of Science of the imperial University of Kyoto. I met also Hiroshi Kawakami who I knew since the conference in Toulouse in 1982 and with whom I will discover the Alpazur oscillator in 1991 [93, 94]. On 31 August-1st September, I traveled to Hiroshima to visit Masayasu Mimura. Discussing with him, I suddenly realized that my vocation in research will be building mathematical models, rather than proving theorems, what I did until today. I turned back to Kyoto on the precise day the Korean Air Lines Flight 007 which was a scheduled flight from New York City to Seoul via Anchorage Alaska, was shot down by a Soviet Su-15 interceptor. All 269 passengers and crew aboard were killed, arousing strong indignation in Japan. The last week of my stay in Japan was devoted to the IUTAM symposium on “turbulence and chaotic phenomena in fluids” (September 5-10) in Kyoto (see Figure 46). During the symposium, Yamaguti

[11] Translated from French

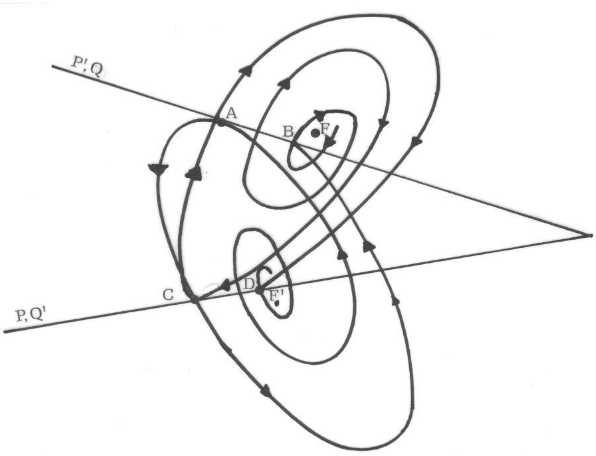


Figure 44. Sketch of a slow-fast construction for the Lorenz attractor.

introduced me to Benoît Mandelbrot who remembered meeting me at Cerisy–la–Salle.

After my first stay in Japan, I could find some grants from C.N.R.S. allowing Ushiki to spend several months every year in Nice between 1984 and 1992. We worked on slow-fast models mimicking the Lorenz attractor. The Lorenz model does not belong to this class of ODE. In order to link this model to slow-fast equations, I proposed an ideal geometrical representation of the Lorenz attractor on a cusp (see Figure 44) [121]. Hiroe Oka and Hiroshi Kokubu, two Ph.D. students of Yamaguti and Ushiki, investigating systematically chaotic attractors modeled by constrained systems, using implicit differential equation (also called generalized vector field), found the set of equations

$$\left\{\begin{array}{l} \epsilon\dot{x} = y - \alpha xz - x^3, \\ \dot{y} = Ax + By + \beta xz, \\ \dot{z} = Cz + \gamma x^2, \end{array}\right. \tag{10}$$

that produces a similar attractor (see Figure 45) to the model I draw by hand. With Ushiki we studied thoroughly equations linked to (10) in this paradigm, using some tools introduced by Argémi: pseudo singular points, fold and co-fold curves, together with others defined by Takens: induced and reduced vector field [196]. Especially, we studied the family of differential equations constrained on a cusp surface (11), similar to (10) [200].

$$\left\{\begin{array}{l} \epsilon\dot{X} = Y + \frac{3}{4}XZ - X^3, \\ \dot{Y} = \tau^2 AX + \tau BY + \frac{3\tau}{4\alpha}\beta XZ, \\ \dot{Z} = \tau CZ - \frac{4\alpha\tau}{3a}\gamma X^2, \end{array}\right. \tag{11}$$

My personal search for a good realistic model for chaos comes to an end with the knowledge of the chaotic electric system by Leon Chua discovered in August 1983 when he was visiting professor in the laboratory of Takashi Matsumoto, at Waseda University, Tokyo [132]. I first presented the results on constrained Lorenz model (11) during the conference on chaotic phenomena at Issy-les-Moulineaux near Paris, on December 3-5, 1986 [125]. During this conference, Leon Chua gave a talk on his

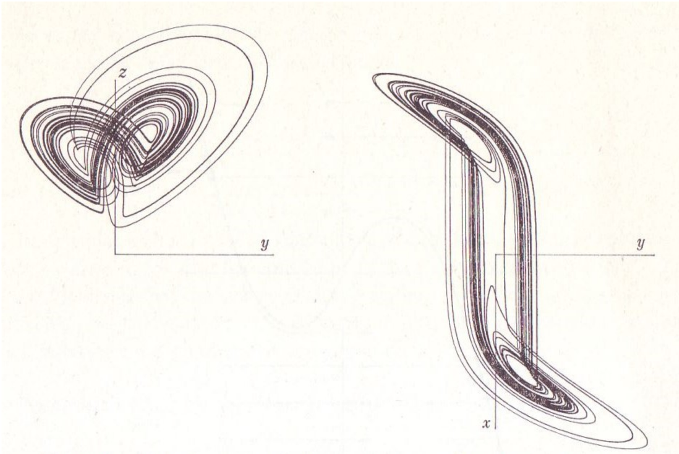


Figure 45. Chaotic attractor produced by the slow-fast system (10). Parameter values: $\epsilon = 0.03$, $A = 0.7$, $B = 0.7$, $C = -1.0$, $a = 0.7$, $\beta = -1.0$ and $\gamma = 1.0$.

Figure 46. Masaya Yamaguti, Shigehiro Ushiki and René Lozi during the IUTAM symposium on "Turbulence and chaotic phenomena in fluids" (September 5-10, 1983) in Kyoto.

Figure 47. Leon Chua and René Lozi on West Lake in Hangzhou, China, in 2014.

electric circuit [44] It was the first time I met him. One year later, on 22 December, traveling in the south of France, he visited me briefly in Nice and invited me to visit him. This was the beginning of a very long cooperation which lasts more than thirty height years with Chua (see Figure 47). With Ushiki, we quickly switched our attention to the Chua circuit, because, from our knowledge, it was the first realistic model for chaos, with only three variables, the minimum number required to produce chaotic solutions.

To conclude this recollection, I would like to mention my relationships with four professors (one Ukranian and three Russians) who made major contributions to chaotic dynamical systems during the decades 1960-80: Vadim Anishenko, Vladimir Belykh (Belykh's attractor), Alexander N. Sharkovsky (Sharkovsky's order), Leonid Shilnikov (homoclinic bifurcations) and my brief encounter with Romen Plykin and Evgenii Sataev.

Before the fall of the Berlin Wall on November 9, 1989, it was uneasy to meet researchers from USSR. Even if it was possible for them to travel to West (M. I. Rabinovich, Ya. G. Sinaï and few other Russians from the Landau Institute, Moscow, attended the New-York conference ten years before). Very often they did not get their visa on time or they had a lack of money to travel, and they missed the conference. Moreover, most of them did not speak English. They published in Russian. It took years for few of their results to reach the West community through translation[12]. A prototypal example of the difficulty to make known their results is the differential treatment of the Li and Yorke theorem "period three implies chaos" published in 1975 [111] and the "Sharkovsky's order" published in an Ukrainian journal in 1964 [186, 187]. In the 70s, the studies of chaotic phenomena were soaring. Many amazing results were published like the Feigenbaum's constants for one-dimensional maps [66], and the Hénon map in dimension two or the Lorenz and the Rössler attractors for ODEs. In many seminars, the result of Li and Yorke was hailed as an important mathematical discovery. It is only two years after, in 1977, that the Sharkovsky's order was published in English by Stefan [194]. After that, this order overshadowed the Li-Yorke theorem which was perceived as a peculiar case of this order, published

[12] for instance *Doklady Mathematics* was a journal consisting of English translations of articles published in Russian.

Figure 48. Alexander Nikolaevich Sharkovskii, Leonid Shilnikov and René Lozi upon their arrival in Honolulu Airport on December 5, 1993.

nine years before. However, it is not exactly the case as Yorke himself reminded me during the banquet of the International Conference on Complex Systems and Applications, in Le Havre on 28 June 2009. I was sited in front of him and near Leon Chua. Yorke said that this order was not giving any result of chaotic properties of continuous function, contrarily to the theorem he proved with Li 34 years before, their paper analyzing a situation in which the sequence of iterates of a map is non-periodic and might be called “chaotic.”

It is during my third one-month visit at Berkeley from November to December 1993, that I have had the opportunity to meet both Shilnikov and Sharkovsky. At this time, Chua was planning the special workshop “Chua’s circuit: chaotic phenomena and applications” during the International Symposium on Nonlinear Theory and its Application (Nolta’93) in Honolulu, December 5-10. As usual, he was a terrific organizer. To get a large audience, he invited in Berkeley many speakers to stay one month in his laboratory and chartered a flight for all of them to Honolulu. Then, early in the morning, Sunday December 5, 1993, with Sharkovsky, Shilnikov, Anishenko (Saratov, Russia), Martin Hasler (Lausanne, Switzerland), Maciej Orgozalek (Krakow, Poland), and many others, we started our trip until to be lodged at the Sheraton Waikiki, Honolulu for one week (see Figure 48)! I did not published with Vadim Anishenko, however, I met him several times after Hawaii, in Warsaw, Nizhny Novgorod, etc., and discussed with him with great pleasure.

Diana Chua, Leon’s wife organized our dispatching in the rooms. I was not smoking, nor Sharkovsky, contrarily to Anishenko and Shilnikov who shared the same room. Therefore, she planned that I must share the room with Sharkovsky. Consequently, we both discussed a lot during the week, even if he was not very fluent in English. This was the beginning of a many long years of cooperation! During the following years, with Bruno Rossetto and Ahmed Aziz-Aloui (my first Ph.D. student, now full professor in Normandy university), we found several grants between 1994 and 2002 to invite Sharkovsky in Toulon, Le Havre and Nice. I published three papers, two with him, Aziz-Alaoui and A. D. Fedorenko from Kiev, and one with him and J. Sousa Ramos from Lisbon. Sharkovsky invited me to the Summer school on dynamical

Figure 49. From left to right, René Lozi, Romen Vasilievich Plykin and Evgenii Sataaev on August 2000, during the Summer school on dynamical system, Katsiveli, Crimea.

system, Katsiveli, Crimea, in August 2000, and to the European Conference on Iteration Theory (ECIT) held in Yalta, Crimea, in September 2008. I worked with him and Sousa Ramos in Lisbon in May 2004. I last met him during the ECIT 2012 in Ponta del Gada, Azores in September 2012, and received his last personal mail few months before is death, on 5 July 2022.

I had the pleasure, during the ten days spent in Katsiveli, to meet two other important figures of the dynamical systems in USSR: Evgenii Sataaev and Romen Vasilievich Plykin (1935-2010) [99], both from Obninsk State Technical University (see Figure 49). The venue of the Conference, the building of the Ukrainian academy of sciences, was standing near the beach, on the Black Sea. The attendants were spending the free long hours in the afternoon, swimming during this hot summer, before the scheduled evening sessions. The banquet took place on a pontoon during the night of August 28th. I strongly disappointed Plykin by not accepting his insistent invitation to dive down with him from the pontoon into the Black Sea, around midnight, after the traditional speeches washed down with vodka!

The year after the Honolulu workshop, Shilnikov visited Christian Mira in Toulouse in 1994. From there, he phoned me to visit my Institute. I arranged a stay for him and his assistant Dimitri Turaev from Nizhny Novgorod, on February 18-23, 1994. Shilnikov gave a talk on 18 Friday. The next day, I drove them to visit Monte-Carlo. Shilnikov absolutely wanted to have a picture of him and me in front of the famous Casino where Anton Tchekov and his friend Ignaty Potapenko lost all their money in 1898. Next, visiting the Oceanographic Museum of Monaco, he told me about his fishing trips in Siberia north of the polar circle. In order to work together we applied twice unsuccessfully for INTAS programs. I never met him again. I attended the conference Dynamics, Bifurcations and Strange Attractors dedicated to his memory, held at the Lobachevsky State University of Nizhny Novgorod, Russia, on July 1-5, 2013, where I met for the first time Vladimir Belykh. I was very impressed to finally meet him after knowing his attractor for decades. Since then, we have become friends and shared our admiration for the songs of Bulat Okudzhava we sang together (see Figure 50).

Fifty years after the beginning of my academic life, I realize that I was incredibly lucky to begin my career as the same time as the emergence of chaos studies in

Figure 50. Vladimir Belykh and René Lozi, singing popular songs of Bulat Okudzhava during the banquet of the Conference NOMA–2017 on June 2017, in Nizhny Novgorod

dynamical systems. I had a great feeling of freedom, being able to discover a new world of extreme richness, as undoubtedly explorers of America had it hundreds of years before me. I had also the privilege to discuss with outstanding researchers, among them five Fields medalists: Thom, Yoccoz, Smale, Alain Connes (within the national committee of C.N.R.S. to which I belonged for nine years) and Pierre-Louis Lions (during seminars in Paris), one Abel prize laureate: Yves Meyer (in the same committee), and many renowned scientists: Belykh, Chua, Hénon, Lorenz, Rössler, Sharkovsky, Shilnikov, Yorke, etc. In this recollection, I have described roughly half of my research life, between 1972 and the beginning of the third millemium. Since the 2000s, I focused my research on concrete applications of chaos theory like pseudorandom number generators [124], cryptography [91], global optimization [84], etc. I explored the new ideas of fractional mappings [97] and fractional derivative [1, 2], and recently built model of Tuberculosis with Saber Elaydi. I also had a thirty year long cooperation with Abul Hassan Siddiqi, president of the Indian Society of Applied and Industrial Mathematics (ISIAM) which I helped him establish in 1990, and Pammy Manchanda Secretary of this society belonging to the The International Council for Industrial and Applied Mathematics (ICIAM) [129, 130]. Since 2013 I have published many paper with Guanrong (Ron) Chen, Editor in Chief of the International Journal of Bifurcation and Chaos, founded by Chua (see Figure 51).

Nowadays, the Belykh, Hénon, Lorenz, Lozi maps, the Chua, Lorenz, Rössler systems and many others are routinely used for many applied purposes. They are living their own life, without the help of their "fathers", like emancipated childrenù[13]. The most amazing fact for me is that, in 1975 and during many years, I tried unsuccessfully to find an efficient algorithm for global optimization. Now, the map I introduced in an entirely different aim is often used within evolutionary algorithms to find global solution of optimization problems involving hundreds of variables [123]. This emphasizes the intimate unity of mathematics.

[13]A strange thing is when I attend a conference in which one lecture uses explicitly this map, and that I speak with young researchers who read my name which is quite uncommon on my badge, they never make the connection between the map and me. I did the same when I was in Oberwolfach with Ersnt Hölder in the room!

Figure 51. Ron Chen and René Lozi, in the south of France in 2013.

I am indebted to the following friends and colleagues who helped to refresh my memory: Ahmed Aziz-Alaoui, Lenore Blum (Carnegie Mellon University, Pittsburgh), Jean Céa, David Chillingworth, Brian Quigley (Head, Engineering & Physical Sciences Division/Mathematics, Statistics & Computer Science Librarian, University of California, Berkeley), Jean Petitot, Paul H. Rabinowitz, Shigehiro Ushiki, Monique and Jeanne Zerner, wife and daughter of Martin Zerner, to Stéphane Junca for shrewd remarks on continuation methods, to Leon Chua for stimulating discussions on the history of nonlinear circuit theory.

11. Conclusion

We often hear that the language used by mathematicians is universal. This means that they write their equations with the same symbols, regardless of the alphabets, ideograms, or other symbols they use to write the text in their theorems. Moreover, they employ the same objects to describe their reasoning: group, functions, topologies, spaces, etc., and the same logic for their proofs. We could infer from these similarities that they think the same way in their lives. A question then arises: are there similarities in the paths to their university careers and the way in which they chose it?

We asked nine experienced, long-career mathematicians specialized in discrete dynamical systems, from five countries (seven if we include their country of origin), to describe how they started their careers long time ago (eight of them are over seventy years old but remain very active). We can point out some similarities and diversities in their backgrounds.

Mathematics was always present from their youth, to varying degrees. It was the main focus of Michał Misiurewicz since childhood. His first encounter with this discipline which he remembered (without having the memory of how old he was then) was counting the cars of a moving train. The urge to count various things remained with him for his whole life until now. By the age of five, when he learned to read, his favorite book was the math textbook for the first grade. Later he was fascinated by the book *Lîlâvatî* by Szczepan Jeleński. In the high school he found a lot of interesting problems in the Hungarian journal *Matematikai Lapok*, that a friend of him was subscribing. He participated in two International Mathematics Olympiads during his high school years (calling them a "great adventure").

Jim Michael Cushing remembered that he was so enamored with mathematics when he was a student in the late 1950s, that he would check out the textbook from the local Carnegie library during the summer recess before he took a maths course in order to teach himself, as best he could, the topic prior to the upcoming school year. In this way he was introduced to not only the standard courses in algebra, but plane geometry (using Euclid's Elements), solid geometry, trigonometry, and single variable calculus. During summer school recesses he would also occupy himself with applications of mathematics (measuring heights of trees using trigonometry, etc.), which was the beginning of a life long love of applied mathematics.

René Lozi indicated that like Misiurewicz, he was counting many things around him in his early childhood. During his preschool year, his teacher did not let him play during recess. She made him do math exercises that the other students were not able to do. When he was 13, one of his cousins, an engineer, introduced him to the base 10 logarithms. He was fascinated and constructed a cardboard slide rule, grading it with his older sister's logarithms chart, and used it at school, much to the astonishment of his classmates and the teacher.

Saber Elaydi sought solace and inspiration within the realm of mathematics in a world plagued by uncertainty and limited resources. The numbers and theorems became his refuge, offering a sanctuary where possibilities knew no boundaries. Armed with unwavering determination and an unquenchable thirst for learning, he delved into the world of mathematics, voraciously consuming every piece of knowledge he could find.

The decision to study mathematics was taken by Laura Gardini when she was 12 years old. There was not a specific event that lead to her decision, she simply loved mathematics and realized that she was happy studying it, and that it was easy for her. At the lyceum she was always the first in mathematics, and at the University in Bologna she was always among the best students in her class.

Vladimir Belykh was a gifted child in all areas and after completing his primary education he was accepted into an elite military boarding school in St. Petersburg. However, a massive reorganization of the Soviet army in 1960 led to the closure of this school. So he graduated from high school in his hometown and was after accepted to Gorky State University where he studied mathematics and physics in the chair of theory of oscillations and automatic control, named of its founding director, Alexander A. Andronov.

However, mathematics was for some in competition with other topics. Eckehard Schöll recalled that as a schoolboy he was interested in a very broad spectrum of subjects, ranging from Latin, English, French, German via History, Music, Art to Mathematics and Physics. He also loved repairing old clocks, building radios, and solving maths puzzles. But when he passed the final school exam (Abitur) with best grades in all subjects in 1970, it was clear that he wanted to study physics at the

University.

Lozi was fascinated by particle physics and dreamed to make researches in theoretical physics, without knowing which courses to attend to reach this goal. However, when graduated with a certificate in fields and particles, he no longer liked this discipline but he discovered numerical analysis.

Galina Strelkova said that during her high school years, her interests were quite broad. She liked mathematics and physics but was very attracted to medicine, she also seriously studied music. She expected that after graduating from school she would certainly continue her studies either at a medical institute or at the Conservatory. However, by the time she graduated from high school, she definitely decided to become a school teacher in mathematics, physics and computer science. She was incredibly fond of mathematics, but to a greater extent not as a beautiful and abstract science, but as an essential tool for describing natural phenomena and events and solving the most interesting physical problems.

Lyudmila Efremova when she was in school not only loved solving difficult mathematical problems, but also made attempts to write music (she also studied at a music school). Therefore, the choice of the further way was very difficult for her. Nevertheless, in the struggle between her two serious hobbies, mathematics and music, mathematics won, and she entered the Faculty of Mechanics and Mathematics of Nizhny Novgorod University.

A second similarity is music that was of great importance to Jim Michael Cushing, Lyudmila Efremova, Eckehard Schöll, and Galina Strelkova. Cushing spent a great deal of time studying piano performance. Upon graduation from high school in 1960 he had various opportunities that included a scholarship offer in piano performance. During his travels on this long career path, he never stopped playing the piano. He even worked with string students and their professors in the University's music department for several years. At school, one of Efremova's compositions for the symphony orchestra was performed at a major concert at her music school. Later she listened lectures by famous musicologists at the Gorky Conservatory on music theory, V.G. Blinova and M.M. Valentinov the main director of the Gorky Opera and Ballet Theater. During his undergraduate studies at the Universities of Stuttgart and Tübingen, Schöll studied special courses in history of arts and musicology and all through his life, he played the piano and sang in choirs, performing many concerts. Strelkova during all her school years seriously studied music (piano and vocals), composed music for poetry, took part in various competitions and became the winner of them several times.

Another key common point appears when reading this memory: for these nine mathematicians, many outstanding researchers had a great influence at the beginning and during their careers. They met them, either in person or by reading their books and several times interacted with them. These mathematicians also interacted with each other, forming an intriguing social network world-wide, across all borders of nationality and languages.

We can find the names of Abraham, Andronov (Andronov-Hopf bifurcation), Anosov (Anosov diffeomorphisms, Anosov flow), Arnold (nominated for the 1974

Fields Medal), Bautin (Bautin bifurcation and Bautin theorem), Dieudonné (one of the founders of the Bourbaki group), Ginzburg (Nobel Prize of physics 2003), Hölder (Hölders' space), Kolmogorov (Kolmogorov or Kolmogorov-Sinai entropy), Leontovich-Andronova, Mazur (Stefan Banach Prize, 1949), Mira, Neimark (Neimark-Saker bifurcation), Novikov (Fields medalist, 1970), Pesin (Pesin theory or theory of non-uniformly hyperbolic dynamical systems), Plykin (Plykin attractor), Sousa Ramos, Saker, Sataev, Shilnikov (Shilnikov bifurcation), Sinai (Abel Prize, 2014), Sharkovsky (Sharkovsky' order), Sobolev (Sobolev's space), Strelcyn, Szlenk (of Szlenk indices fame), Thom (Fields medalist, 1958) and many other scholars in the recollections of Vladimir Belykh, Lyudmila Efremova, Laura Gardini, René Lozi, and Michał Misiurewicz.

For example, René Lozi said that in 1969-70, he was a student of Jean Alexandre Dieudonné at the university of Nice. His Ph.D. advisor Jean Céa recalled him recently the visit of Sobolev who was fluent in French and knew a lot about the history of France, at the first colloquium on numerical analysis he had organized in Paimpol (France) in 1967. In 1976, at Oberwolfach he listened a conference of Ersnt Hölder, the son of Otto Hölder, the mathematician whose name was given to the Hölder spaces.

Vladimir Belykh followed the teachings of Yuri I. Neimark, Dmitry A. Gudkov, Nickolai A. Fufaev, and Nickolai A. Zheleztsov, the brilliant team of Andronov's disciples. He was accepted into the Division of Differential Equations headed by Evgeniya Alexandrovna Andronova-Leontovich, Andronov's spouse and close collaborator and a sister of another celebrated Soviet physicist, Mikhail A. Leontovich of Moscow State University (Lenin Prize 1958). He became friend with Leonid P. Shilnikov who worked in the same division, and who one day explained to him his famous theorem on the bifurcation of a homoclinic orbit of a saddle focus. In 1974 Belykh shared the prestigious Lenin Komsomol Prize (a Soviet science award for young researchers under 33) with his colleagues Vladimir D. Shalfeev and Valery P. Ponomarenko. In 1976, Valentin (Valya) Afraimovich, a Shilnikov's pupil, explained to him the action of the discontinuous Poincaré return map of the Lorenz attractor he constructed in a qualitative, implicit form with Shilnikov and Vadim Bykov. Belykh realized soon that his model of a discrete-time phase-locked loop (PLL) in the case of a piecewise linear nonlinearity was comparable to their map and showed this model with a chaotic attractor to them. Afraimovich said: "Here is the Belykh attractor" and later informed Sinai about this new object of potential interest for the ergodic theory. Belykh also gave a talk at Anosov's seminar, and Dmitry V. Anosov's pupils (Yakov Pesin, Evgeny Sataev, and Nikolai Chernov) began their analysis of the ergodic properties of such map. Anosov strongly recommended him to include the description of this map and its detailed analysis in his second doctoral thesis. Later, in 1983, Anosov served as an opponent for his "Doctor of Sciences" degree thesis. The same year he moved to the Institute of Water Transport Engineers in Nizhny Novgorod, where he took the position of its Mathematics Department's head, which became available after the retirement of Nikolaï N. Bautin.

Lyudmila Efremova listened to the lectures of Sergueï Petrovitch Novikov, Yakov Grigorievitch Sinai, and physicists Vitaly Lazarevitch Ginzburg, Andrei Viktorovich Gaponov-Grekhov (Ph. D. student of Andronov) in Nizhny Novgorod. Her survey on skew products of maps of an interval in dimension two was published in *Russian Mathematical Surveys* and recommended to Sinai by Professor Stepin. In 2009 and again in 2011 D.V. Anosov was the scientific supervisor of her internship at the Steklov Mathematical Institute of the Russian Academy of Sciences. She interacted with him until the beginning of 2014, and in August 2014 when he died. In the autumn of 2013, although he was feeling unwell, he recommended her to professor Stepin who helped her until the defense of her Doctor of Sciences Thesis in 2018 that she defended at the Institute of Information Transmission Problems of the Russian Academy of Sciences the concil of which was chaired by Professor Sinai. The official referees were Belykh (one of the most prominent representatives of the school of Andronov), A. Zhirov (a former Ph. D. student of Plykin) and the famous Russian topologist S. Bogatyi.

Michał Misiurewicz described the origin of the dynamical systems in Warsaw, when Stanisław Mazur sent two young Warsaw mathematicians, Wiesław Szlenk and Karol Krzyżewski to Andrey Kolmogorov who adressed them to Sinai. They returned to Warsaw after learning the basics of the dynamical systems. Sinai followed them and gave a cycle of lectures in ergodic theory. Another member of this Warsaw group was Jan Maria Strelcyn, who was taking notes from Sinai's lectures, that were later available as a booklet. Around 1969 Strelcyn left Poland for France, but Szlenk, and later also Krzyżewski, started to build a group in this new area of mathematics. The first two students from the new generation were Michał Krych and Misiurewicz. Next students that joined were Feliks Przytycki and Maciej Wojtkowski, and after them many others. In 1977 Misiurewicz wrote with Szlenk a paper, where they proved some basic results in the theory of interval maps.

In 1994, during a visit to the University of Southern California, Saber Elaydi met Robert Sacker (the author of the famed Neimark–Sacker bifurcation) and started working jointly. He met Jim Yorke at the fifth ICDEA, which was held in Temuco, Chili, in January 2000.

In 1975, Belykh with his Ph. D. student Vladimir I. Nekorkin, rigorously proved the existence of Shilnikov's chaos in a three-dimensional autonomous phase system.

In 1994, at Katsiveli, in Crimea, at the conference "Differential Equations, Bifurcations and Chaos" organized by Yuri and Vladimir Maistrenko, and Iryna Sushko (Ph. D. student of Y. Maistrenko and Alexander Sharkovsky) Laura Gardini met Leonid Shilnikov and his son Andrey in 1994, and soon she became friends with them. They visited her in 1995 in Urbino and asked her to explain her result on homoclinic bifurcations of expanding cycles in $\mathbb{R}^n$. In September 1996 the famous *Blue sky catastrophe model* was presented (for the first time outside Nizhny Novgorod) by Leonid Shilnikov at the *European Conference on Iteration Theory* that she organized in Urbino.

Lozi met Leonid Shilnikov and Alexander Sharkovsky in the Leon Chua' laboratory in San Francisco in 1993. Shilnikov after having worked with Christian Mira in Toulouse in 1994 visited him with Dimitri Turaev in the university of Nice in 1994 and gave a talk in the mathematics department.

The study of some books by Thom, Zeeman, Postom and Stewart, Collet and Eckmann, Guckenheimer and Holmes decided Laura Gardini to spend her academic life in the research of dynamical sytems. Thom accepted to review the "Thèse d'État" of Lozi in 1983 being interested by the ontology of the slow-fast metaphoric model of the Belousov-Zhabotinsky reaction proposed in it. He suggested to Ruelle to invite him to give a talk at I.H.E.S the same year. In 1986, Thom reviewed the Lozi's memoir for the recently created Habilitation diploma and assisted him many times in his applications to higher position in the university. In 1990, Lozi was one of the height mathematicians (with M. Chaperon, A. Chenciner, J. Martinet, J.P. Ramis, P. Moussa, R. Moussu, F. Pham,) invited to pay tribute to Thom on his retirement during the "Rencontre entre mathématiciens et physiciens théoriciens" in Strasbourg.

Belykh met first Alexander N. Sharkovsky in 1965 when he presented the proof of his famous numbers ordering at a research seminar run by Neimark in Gorki. In 1972 Sharkovsky agreed to be external reviewer of his Ph.D. thesis initiated in 1968 under the guidance of Lyudmila N. Belyustina at NII PMK's Division of Dynamical Systems and Control Theory, headed by Neimark.

N. F. Otrokov and his former graduate student Dr. Rakhmankulov were the scientific supervisors of Lyudmila Efremova. N. F. Otrokov was a major specialist in the theory of limit cycles of differential equations and he defended his PhD thesis under Andronov's supervision in 1940. When she was a 2nd year student, Rakhmankulov invited her to read the Sharkovsky's paper. Now after many years she can say that her first impression of this theorem determined her scientific biography. Throughout her work in mathematics, from time to time she returns to various generalizations of it. In 1977 when she was a Ph.D. student of the 2nd year, Sharkovsky under the request of Otrokov invited her to present her research at his seminar at the Institute of Mathematics of the Academy of Sciences of Ukraine. After that, she has had the opportunity to visit several times Sharkovsky in Kiev to present and discuss new results. She had also many interesting discussions with Professors Plykin and Szlenk. She recalls, with great warmth, the very lively discussions of her research with Belykh who was with Sharkovsky, the official referees of her Ph. D. thesis.

After meeting Alexander Sharkovsky in 1993, Lozi began a long collaboration with him. Between 1994 and 2002 he found several grants to invite Sharkovsky in France (in Nice, Le Havre and Toulon). He published three papers, two with him and A. Aziz–Alaoui from Le Havre and A. D. Fedorenko from Kiev, and one with him and J. Sousa Ramos from Lisbon. Lozi was invited by Sharkovsky to the Summer school on dynamical system, Katsiveli, Crimea, in August 2000, where he met Romen Plykin and Evgenii Sataev (both from Obninsk State Technical University), and to the European Conference on Iteration Theory (ECIT) held in Yalta, Crimea, in September 2008 . He worked with Sharkovsky and Sousa Ramos in Lisbon in May 2004. He last met him during the ECIT 2012 in Ponta del Gada, Azores in September 2012, and received his last personal mail few months before is death, on 5 July 2022.

In the spring 1991 Laura Gardini was invited by Christian Mira to visit him in Toulouse. That started her long time friendship and collaboration with him. The same year in September, he invited her to participate in the *European Conference on Iteration Theory* in Lisbon, organized by Sousa Ramos where she met Jaroslav Smital, Sergei Kolyada, Ludomir Snoha, Francisco Balibrea and Jaume Llibre. In 1992, Laura Gardini presented some results on particular homoclinic bifurcations at two more conferences, one in Germany and one in Italy where, besides Christian Mira, also Ralph Abraham was invited, and with him she also started a long friendship and collaboration. He encouraged her research work, and was fascinated by the dynamics of noninvertible maps. He was visiting professor in Urbino for some periods (six times from 1992 to 2000). She also worked closely with Christos Frouzakis and Ioannis Kevrekidis whom she met in Minneapolis in March 1995.

Not only others outstanding mathematicians like Anishchenko, Bownds, Chillingworth, Conley, Martin, Yorke, Milne–Thomson had a great influence on the authors of this article, there are also the astronomer Hénon, biologists (Hájek, Levin), ecologist (MacArthur, May, Rosenzweig), physicists (Chua, Haken, Helleman, Heisenberg, Kocarev, Martin, Ruelle, Shaw) and meteorologist (Lorenz) and many others who inspired the authors of this article.

In 1976 Michel Hénon published a paper about his attractor. After spending about half a year on trying to prove that it is really an attractor, Misiurewicz decided that temporarily he will investigate interval maps and only when he gain substantial knowledge of them, he will return to the Hénon's attractor. In fact, he remained stuck with one-dimensional dynamics. In 1977, spurred by a presentation by David Ruelle of the Hénon attractor, Lozi proposed a piecewise linear model of this attractor.

In 1983, under the guidance of Otomar Hajek (von Humboldt award at TU Darmstadt), Saber Elaydi wrote several papers on the dichotomy and trichotomy of nonautonomous differential equations. With the help of his friend Gerry Ladas, he established the Journal of Difference Equations and Applications (JDEA) in 1994. They organized the first international conference (ICDEA I) on difference equations in San Antonio, Texas, in May 1994, which has since been an annual conference. Elaydi met Jim Yorke at the fifth ICDEA, which was held in Temuco, Chili, in January 2000. In 2001, with Bernd Aulbach they created the International Society of Difference Equations (ISDE). In the spring of 1995, he suggested to the Trinity University to invite Lord Robert May (Copley medalist, 2007) to give a talk as part of the "Distinguished Science Lectures Series". The day after the conference, he had the pleasure of asking him several questions about chaos and the evolution of species, and discussed with him his new book project on discrete dynamical systems.

Jim Michael Cushing recalls that his Ph.D. dissertation advisor was Monroe H. Martin (Guggenheim Fellowship 1959), who at the time was the director of the Institute of Fluid Dynamics and former chair of the Mathematics Department at Maryland. Martin took on dissertation students only one at a time (one of whom, before Cushing, was Simon A. Levin (National Medal of Science – 2016), who is currently the director of the Center for BioComplexity at Princeton University). The years of Cushing at Maryland (1964-1968) were mathematically quite exciting and fruitful, in large part because of numerous high level faculty members and student classmates (one of whom was the celebrated James Yorke (Japan Prize 2003), who went on to coin the word "chaos" in dynamical systems). In the Fall of 1968 he took up a position of assistant professor of mathematics at the University of Arizona in Tucson where was the celebrated fluid dynamicist Louis Melville Milne–Thomson (Milne-Thomson circle theorem) and some of his students. In 1970 having a post-doctoral position at the IBM Thomas J. Watson Research Center, in Yorktown Heights he attained a renewed interest in dynamic stability, mainly through stimulating discussions with Charles Conley (of Conley index fame), who was his office mate for a while. Back in Arizona in 1976, he began a collaboration with John

Bownds (who introduced the concept of ROW (Rest of the World) to the search and rescue community and received the National Award from the National Association For Search And Rescue in 1991), on stability theory for integral and integro–differential equations, however, his growing involvement with population and ecological dynamics was greatly enhanced by the foundation of two programs in 1976 at the University of Arizona: the Interdisciplinary Program in Applied Mathematics and the Department of Ecology and Evolutionary Biology (EEB) where he met Michael Rosenzweig (of predator-prey "paradox of enrichment" fame) and William Schaffer, both students of Robert MacArthur at Princeton University who is considered a founding father of ecology and evolutionary biology. In 1989 he met biologist Robert Costantino, of the University of Rhode Island. The result was the creation of a team of four interdisciplinary researchers (the Beatle Team) consisting of Bob Costantino, Bob Desharnais and him together Brian Dennis, an ecologist and statistician, later joined by Shandelle Henson and Aaron King. His most recent interdisciplinary collaboration have been with Shandelle Henson and her husband, James Hayward, a field ecologist who has worked and gathered an immense amount of data on marine animals and birds on Protection Island, a United States Natural Wildlife Refuge managed by the US fish & Wildlife Service.

Eckehard Schöll in Tübingen in 1974 in Mathematics seminars by Rainer Nagel, studied ergodic theory, functional analysis, von Neumann algebras and C^* algebras. Werner Güttinger, introduced a course on nonlinear dynamical systems, where he made his first contact with this field. Harald Stumpf became his first mentor by offering him a Master Thesis (called Diploma Thesis at that time) in theoretical semiconductor physics. In the same year he heard in a Physics Colloquium a lecture by Werner Heisenberg (Nobel Prize 1932) on his unified field theory. He was also very impressed by a talk by David Ruelle (Holweck Prize 1993, who coined the term "strange attractor" with Floris Takens in 1970) at the mathematically oriented "Rencontre entre mathématiciens et physiciens théoriciens" in Strasbourg in 1976. The same year he traveled to Britain and started to work with Peter T. Landsberg in Southampton and started to work with him who became a mentor and a friend to whom he owes a lot. At Southampton he attended also lectures by David Chillingworth who came from the Warwick dynamical systems group. Hermann Haken (Max Born Medal and Prize 1976) who had created the field of Synergetics which deals with these nonlinear systems far from thermodynamic equilibrium and their universal features became very important in his scientific career when he invited him to Stuttgart in 1983 to give a seminar on his work on nonequilibrium phase transitions and self-organization in semiconductors. In 1986 he attended for the first time Dynamics Days in Enschede, Netherlands, a conference series which was founded by Robert Helleman in 1980, and which is organized annually up to the present day. He has attended 23 editions of this series. In 2017 he received an Honorary Doctorate from Saratov State University, Russia, as a result of his very fruitful and active collaboration with the Saratov group of Vadim Anishchenko and Galina Strelkova.

In 1983 Belykh met Vadim Anishchenko (1943-2020, Alexander von Humboldt Research Award 1999) from Saratov State University at a conference on the Oka River, and they quickly became life-long friends. Lozi met for the first time Anishchenko at the International Symposium on Nonlinear Theory and its Application (Nolta'93) in Honolulu. He never published with him, however, he met him again several times after Hawaii, in Krakow, etc. and discussed with him with great pleasure.

Galina Strelkova was a first–year physics student at the Department of Radiophysics (since 1995 it is the Radiophysics and Nonlinear Dynamics Department) of Saratov State University in the early 90s. The Department was headed by Anishchenko who was also a scientific supervisor of a just organized Nonlinear Dynamics Laboratory. When she graduated he invited her to continue her postgraduate studies under his supervision, asking her to understand the properties of chaotic attractors from a physical point of view. She was able to define the fundamental differences in the properties of quasihyperbolic (like Lozi, Belykh, and Lorenz) and nonhyperbolic attractors (like Hénon map, the cubic map, the logistic map, the Rössler system, the Anishchenko-Astakhov oscillator, the Chua's oscillator). The main model systems under study were the Lozi and the Hénon maps. After defending her Ph. D. thesis she was invited several times by Professor Jürgen Kurths for research visits in his Nonlinear Dynamics Working Group in Potsdam University. Her professional career was inextricably linked with the Department of Radiophysics and Nonlinear Dynamics of Saratov. Thanks to the initiated collaboration and the activities of Schöll, Professor of the Technical University of Berlin, this department was involved in Collaborative Research Center (CRC) SFB 910 (2011-2022): Control of Self-Organizing Nonlinear Systems, in the framework of the first Russian Project in a German CRC. Anishchenko and Tatiana Vadivasova were Principal Investigators (PIs) of the Project and since 2019 she became a PI too. After the death of Professor Vadim Anishchenko on November 30, 2020, she headed this department.

Electronics research inspired several authors of this recollection. As part of the European-Soviet scientific exchange program, Belykh visited the Electronics Laboratory at the Danish Technical University (DTU) in Lyngby, during the 1975-1976 academic year where he worked with George Bruun and Orla Christensen, two professors of electronics, studying discrete-time digital PLLs. In the meantime, two DTU experimental physicists, Niels F. Pedersen and Ole H. Soerensen, found out that he was an expert in the dynamics of pendulum equations and asked him to explain a random behavior of the Josephson junctions' current density-voltage $J - V$ curves they obtained experimentally. The research interests of Belykh having later broadened to the dynamics of networks and synchronization led to long-term collaborations with Erik Mosekilde, a physics professor at DTU, and Martin Hasler, an engineering professor at École Polytechnique Fédérale de Lausanne (EPFL).

The first talks which Schöll gave at International Conferences were in 1978 at the International Conference on the Physics of Semiconductors (ICPS) in Edinburgh, in 1980 at the German Physical Society (DPG) Annual Spring Meeting in Münster, and in 1981 at the International Conference on Hot Carriers in Semiconductors (HCIS) in Montpellier, France. At this last conference he met Melvin P. Shaw from Wayne State University, Detroit, author of famous papers and a book on semiconductor instabilities and in particular of a book on the Gunn effect. He visited him for a year in 1983/84 at the Department of Electrical Engineering, and became a coauthor of the book *The Physics of Instabilities in Solid State Electron Devices* finally published in 1992.

After meeting Leon O. Chua at a conference in Crimea, Belykh spent a while at UC Berkeley working with him, Chai Wah Wu, and Ljupco Kocarev. Lozi met Chua near Paris in 1986 three year after the discovery by this professor of electronics of his famous double scroll circuit. One year later, on 22 December, traveling in the south of France, Chua visited him in Nice. This was the beginning of a very long cooperation between them which lasts since more than 38 years. Lozi had also the privilege to discuss with outstanding researchers, among them five Fields medalists: Thom, Yoccoz, Smale, Alain Connes (within the national committee of C.N.R.S. to which he belonged for nine years) and Pierre-Louis Lions (during seminars in Paris), one Abel prize laureate: Yves Meyer (in the same committee), and many renowned scientists: Belykh, Chua, Hénon, Lorenz, Rössler, Sharkovsky, Shilnikov, Yorke, etc. Recently he built model of tuberculosis with Saber Elaydi. he also had a thirty year long cooperation with late Abul Hassan Siddiqi, president of the Indian Society of Applied and Industrial Mathematics (ISIAM). Since 2013 he has published many paper with Guanrong (Ron) Chen, Editor in Chief of the International Journal of Bifurcation and Chaos, founded by Chua.

From this non exhaustive summary of the interactions of the authors of this article with dozens of researcher, we can highlight that Science is very international and conclude like Eckehard Schöll "I am happy to be part of this international family of physicists and mathematicians. This starts with student exchanges with foreign countries, then, international collaborations with colleagues all over the globe, and conference all over the world, where this family meets, form a network." No mathematician, even if he is gifted in this discipline, can flourish alone. To assert his talent, he must be guided and then cooperate with many other researchers.

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