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# QUADRUPLE INEQUALITIES: BETWEEN CAUCHY–SCHWARZ AND TRIANGLE

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*Abstract.* We prove a set of inequalities that interpolate the Cauchy–Schwarz inequality and the triangle inequality. Every nondecreasing, convex function with a concave derivative induces such an inequality. They hold in any metric space that satisfies a metric version of the Cauchy–Schwarz inequality, including all CAT(0) spaces and, in particular, all Euclidean spaces. Because these inequalities establish relations between the six distances of four points, we call them quadruple inequalities. In this context, we introduce the quadruple constant — a real number that quantifies the distortion of the Cauchy–Schwarz inequality by a given function. Additionally, for inner product spaces, we prove an alternative, more symmetric version of the quadruple inequalities, which generalizes the parallelogram law.

## 1. Introduction

### 1.1. Relating Cauchy–Schwarz and triangle

The Cauchy–Schwarz inequality states that in any inner product space $(V,\langle\cdot,\cdot\rangle)$, we have

$$\langle u,v\rangle \leqslant \|u\|\|v\| \tag{1}$$

for all $u,v\in V$, where $\|u\| = \sqrt{\langle u,u\rangle}$. In any metric space $(\mathcal{Q},d)$ the triangle inequality

$$\overline{y,z} \leqslant \overline{y,p} + \overline{p,z} \tag{2}$$

is true for all $p,y,z\in\mathcal{Q}$, where we use the short notation $\overline{y,z} := d(y,z)$. These two inequalities can be connected as follows: Consider the inequality

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \leqslant L_\tau\,\overline{q,p}\,\tau'(\overline{y,z}) \tag{3}$$

for $y,z,q,p\in\mathcal{Q}$, a differentiable function $\tau\colon [0,\infty)\to\mathbb{R}$ with derivative $\tau'$, and a constant $L_\tau\in[0,\infty)$. We call (3) a *quadruple inequality* [19] as it establishes a relationship between the six distances among four points, see Figure 1. If we plug the identity $\tau=\tau_1 := (x\mapsto x)$ and $L_\tau = 2$ into (3), we obtain

$$\overline{y,q} - \overline{y,p} - \overline{z,q} + \overline{z,p} \leqslant 2\,\overline{q,p}\,. \tag{4}$$

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The triangle inequality (2) implies (4). Furthermore, in a symmetric distance space $(\mathcal{Q}, d)$ [8], where $d$ does not necessarily fulfill the triangle inequality, (4) also implies (2) by setting $z = q$. Next, let us evaluate (3) with $\tau = \tau_2 := (x \mapsto x^2)$ and $L_\tau = 1$. We get

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leqslant 2\,\overline{q,p}\,\overline{y,z}\,. \tag{5}$$

If we assume that the metric space $(\mathcal{Q}, d)$ is induced by an inner product space $(V, \langle \cdot, \cdot \rangle)$, i.e., $\mathcal{Q} = V$ and $d(q,p) = \|q - p\|$, then (5) becomes

$$2\,\langle q - p\,, z - y\rangle \leqslant 2\|q-p\|\|y-z\|\,. \tag{6}$$

Thus, in this case, (5) is equivalent to (1). Hence, we can consider (5) to be a generalization of the Cauchy–Schwarz inequality to metric spaces. Equation (5) is not true in every metric space. But it plays an important role in the study of geodesic metric spaces. A geodesic metric space is a metric space such that any two points $q, p \in \mathcal{Q}$ can be joined by a geodesic, i.e., a curve of length $\overline{q,p}$. Equation (5) is well-known to hold in non-positively curved geodesic spaces, which are called $\mathsf{CAT}(0)$ spaces or, if they are complete, Hadamard spaces or global NPC spaces. In these spaces, (5) is also known as *four point cosq condition* [2] or *Reshetnyak's Quadruple Comparison* [23, Proposition 2.4]. Furthermore, a geodesic space is $\mathsf{CAT}(0)$ if it fulfills (5) [2, Theorem 1].

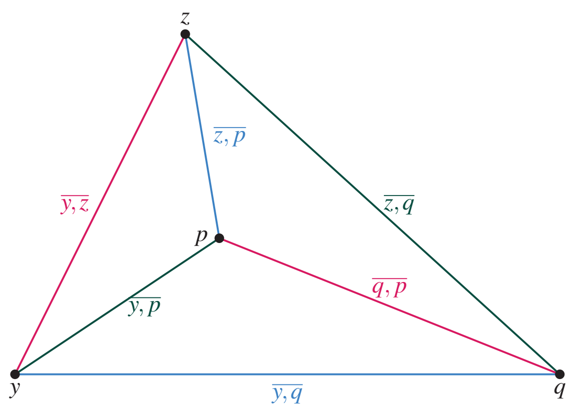


Figure 1: *Four points and their six distances.*

## 1.2. Contributions

The functions $\tau_1, \tau_2$ are both nondecreasing, convex, and have a concave derivative. They can be considered as edge cases of all functions with these properties: As a linear function, $\tau_1$ can be thought of as "least convex" of all convex functions. Similarly, $\tau_2$, which has a linear and strictly increasing derivative, is a "most convex" function among all functions with a concave derivative. As our main result, we show that in all metric spaces with the property (5), inequality (3) is true for all functions "between" $\tau_1$ and $\tau_2$, i.e., for all nondecreasing, convex functions with a concave derivative. In this sense, we "interpolate" the triangle and the Cauchy–Schwarz inequality.

Denote by $\mathscr{S}$ the set of all nondecreasing, convex, and differentiable functions $\tau\colon [0,\infty) \to \mathbb{R}$ with concave derivative.

THEOREM 1. *Let $(\mathscr{Q},d)$ be a metric space. Let $y,z,q,p \in \mathscr{Q}$. Assume*

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leqslant 2\,\overline{q,p}\,\overline{y,z}\,.$$

*Let $\tau \in \mathscr{S}$. Then*

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \leqslant 2\,\overline{q,p}\,\tau'(\overline{y,z})\,. \tag{7}$$

We emphasize that (5) as condition in Theorem 1 is only required for the specific four points $y,z,q,p \in \mathscr{Q}$ in the given order.

As the constant 2 in (4) is optimal (set $y = p$ and $z = q$), we see that the 2 on the right-hand side of (7) is optimal as a universal constant not depending on $\tau$. Furthermore, if this constant is allowed to depend on $\tau$, we show that it must be at least 1 for all non-constant $\tau \in \mathscr{S}$:

Let $\tau\colon [0,\infty) \to \mathbb{R}$ and $L_\tau \in [0,\infty]$. Assume for any metric space $(\mathscr{Q},d)$ and points $y,z,q,p \in \mathscr{Q}$ that fulfill (5), we also have (3). If $L_\tau$ is the smallest value with this property, we call $L_\tau$ the *quadruple constant* of $\tau$.

THEOREM 2. *Let $\tau \in \mathscr{S}$. Assume $\tau$ is not constant. Then the quadruple constant of $\tau$ fulfills $L_\tau \in [1,2]$.*

Note that $L_{\tau_2} = 1$ and $\tau = \tau_2$ in (3) yields the Cauchy–Schwarz inequality. Hence, we can interpret the quadruple constant $L_\tau$ as a factor describing the distortion of the Cauchy–Schwarz inequality induced by applying $\tau$ instead of $\tau_2$.

Let $\mathscr{S}_0$ be the set of functions in $\mathscr{S}$ with $\tau(0) = 0$. For $\tau \in \mathscr{S}_0$, the right-hand side of (7) can be bounded by $2\tau(\overline{q,p}) + 2\tau(\overline{y,z})$, see Corollary 1. In inner product spaces, we derive a stronger upper bound:

THEOREM 3. *Let $(V, \langle\cdot\,,\cdot\rangle)$ be an inner product space with induced metric $d$. Let $y,z,q,p \in V$. Let $\tau \in \mathscr{S}_0$. Then*

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \leqslant \tau(\overline{q,p}) + \tau(\overline{y,z})\,. \tag{8}$$

This can be viewed as a generalization of the *parallelogram law* in inner product spaces, $\|u+v\|^2 + \|u-v\|^2 \leqslant 2\|u\|^2 + 2\|v\|^2$ for $u,v \in V$, as it implies

$$\tau(\|u+v\|) + \tau(\|u-v\|) \leqslant 2\tau(\|u\|) + 2\tau(\|v\|) \tag{9}$$

for $\tau \in \mathscr{S}_0$, see Corollary 3.

### 1.3. Related literature

For a history of the Cauchy–Schwarz inequality and many of its extension, [22] is highly recommended. The book [8] is an excellent reference for metric related concepts.

#### 1.3.1. Convex analysis

Theorem 3 is related to *Karamata's inequality* [17]: Let $f\colon \mathbb{R}\to\mathbb{R}$ be a convex and nondecreasing function. Let $a_1,\dots,a_n,b_1\dots,b_n\in\mathbb{R}$ with

$$\sum_{i=1}^{k} a_i \leqslant \sum_{i=1}^{k} b_i \tag{10}$$

for $k=1,\dots,n$. Then

$$\sum_{i=1}^{n} f(a_i) \leqslant \sum_{i=1}^{n} f(b_i)\,. \tag{11}$$

If we set $f=\tau$, $n=4$, $a_1=\overline{y,q}$, $a_2=\overline{z,p}$, $a_3=a_4=0$, $b_1=\overline{q,p}$, $b_2=\overline{y,z}$, $b_3=\overline{y,p}$, $b_4=\overline{z,q}$, then Karamata's inequality proves Theorem 3 for configurations of distances that fulfill (10). But this does not cover all cases.

#### 1.3.2. Quadruple inequality

Theorem 1 extends [19, Theorem 3 and Appendix E], which imply the following:

PROPOSITION 1. *Let* $\alpha\in(0,\infty)$. *Define* $\tau_\alpha\colon[0,\infty)\to\mathbb{R}, x\mapsto x^\alpha$. *Then the quadruple constant of* $\tau_\alpha$ *is*

$$L_{\tau_\alpha}=\begin{cases}2^{2-\alpha} & \textit{if } \alpha\in[1,2]\,,\\ \infty & \textit{otherwise}\,.\end{cases} \tag{12}$$

Note that $\tau_\alpha\in\mathscr{S}$ if and only if $\alpha\in[1,2]$. The proof of Theorem 1 requires new ideas compared to the one of [19, Theorem 3], e.g., we cannot take derivatives with respect to $\alpha$. Our generalization to all functions $\tau\in\mathscr{S}$ is less precise in the quadruple constant: Theorem 1 applied to $\tau_\alpha$ yields only the upper bound $L_{\tau_\alpha}\leqslant 2$. But $L_{\tau_\alpha}=2^{2-\alpha}<2$ for $\alpha>1$.

#### 1.3.3. Metric geometry

Aside from $\mathsf{CAT}(0)$ spaces briefly discussed above, some further ideas in metric geometry seem relevant in the context of quadruple inequalities.

A function $\varphi\colon[0,\infty)\to[0,\infty)$ is called *metric preserving*, if $\varphi\circ d$ is a metric for any metric space $(\mathscr{Q},d)$. See [7] for an overview. As the quadruple inequality (3) with $\tau=\tau_2$ is a condition for Theorem 1, we may think of the main result as stating that a Cauchy–Schwarz-like inequality is persevered under transformation with $\tau$. But note that the right-hand side of (3) is not written in terms of $\tau\circ d$.

A metric space $(\mathscr{Q},d)$ has the *Euclidean k-point property* [3, Definition 50.1] if any $k$-tuple of points in $\mathscr{Q}$ has an isometric embedding in the Euclidean space $\mathbb{R}^{k-1}$. If $(\mathscr{Q},d)$ has the Euclidean 4-point property, then (5) is fulfilled. For $\gamma \in (0,\infty)$, let $\varphi_\gamma(x) := x^\gamma$. This function is metric preserving for $\gamma \leqslant 1$. According to [3, Theorem 52.1], $(\mathscr{Q},\varphi_\gamma \circ d)$ has the Euclidean 4-point property for all $\gamma \leqslant 1/2$. Furthermore, $\gamma = 1/2$ is the largest exponent with this property. Thus, $(\mathscr{Q},\varphi_\gamma \circ d)$, fulfills (5) for $\gamma \in (0,1/2]$. In particular,

$$\overline{y,q}^{2\gamma} - \overline{y,p}^{2\gamma} - \overline{y,p}^{2\gamma} + \overline{y,p}^{2\gamma} \leqslant 2\,\overline{q,p}^{\gamma}\,\overline{y,z}^{\gamma}. \tag{13}$$

As $\tilde{d} = d^{2\gamma}$ is a metric — $x \mapsto x^{2\gamma}$ is metric preserving for $\gamma \in (0,1/2]$ — we obtain from (4),

$$\overline{y,q}^{2\gamma} - \overline{y,p}^{2\gamma} - \overline{y,p}^{2\gamma} + \overline{y,p}^{2\gamma} \leqslant 2\min\left(\overline{q,p}^{2\gamma}, \overline{y,z}^{2\gamma}\right), \tag{14}$$

which also implies (13).

The Euclidean 4-point property can be weakened for $\mathsf{CAT}(0)$ spaces. A metric space $(\mathscr{Q},d)$ fulfills the $\mathsf{CAT}(0)$ 4-point condition [4, Definition II.1.10] if, for all $y,z,q,p \in \mathscr{Q}$, there are $\bar{y},\bar{z},\bar{q},\bar{p} \in \mathbb{R}^2$ such that

$$\overline{y,q} = \|\bar{y}-\bar{q}\|, \qquad \overline{y,p} = \|\bar{y}-\bar{p}\|, \qquad \overline{z,q} = \|\bar{z}-\bar{q}\|, \qquad \overline{z,p} = \|\bar{z}-\bar{p}\|,$$
$$\overline{q,p} \leqslant \|\bar{q}-\bar{p}\|, \qquad \overline{y,z} \leqslant \|\bar{y}-\bar{z}\|.$$

Every $\mathsf{CAT}(0)$ space fulfills the $\mathsf{CAT}(0)$ four-point condition, see [18] or [4, Proposition II.1.11].

Another famous 4-point property is *Ptolemy's inequality*: A metric space $(\mathscr{Q},d)$ is called *Ptolemaic* if, for all $y,z,q,p \in \mathscr{Q}$, we have

$$\overline{y,q}\,\overline{z,p} + \overline{y,p}\,\overline{z,q} \leqslant \overline{q,p}\,\overline{y,z}. \tag{15}$$

Every inner product space is Ptolemaic. If a normed vector space is Ptolemaic, then it is an inner product space. All $\mathsf{CAT}(0)$ spaces are Ptolemaic. A complete Riemannian manifold is Ptolemaic if and only if it is $\mathsf{CAT}(0)$ [5, Theorem 1.1]. Each geodesic metric space satisfying the $\tau_2$-quadruple inequality is Ptolemaic, but a geodesic Ptolemaic metric space is not necessarily $\mathsf{CAT}(0)$ [13, 2].

Strongly related to Theorem 3 is the concept of roundness of a metric space: A value $\alpha \in (0,\infty)$ is called *roundness exponent* of a metric space $(\mathscr{Q},d)$ if, for all $y,z,q,p \in \mathscr{Q}$,

$$\overline{y,q}^{\alpha} - \overline{y,p}^{\alpha} - \overline{z,q}^{\alpha} + \overline{z,p}^{\alpha} \leqslant \overline{q,p}^{\alpha} + \overline{y,z}^{\alpha}. \tag{16}$$

Let $R = R(\mathscr{Q},d)$ be the set of all roundness exponents of $(\mathscr{Q},d)$. The *roundness* $r = r(\mathscr{Q},d)$ of $(\mathscr{Q},d)$ is the supremum of the roundness exponents $r := \sup R$. By the triangle inequality and the metric preserving property of $(x \mapsto x^\alpha)$ for $\alpha \in (0,1]$, we have $(0,1] \subset R$ for all metric spaces. The function spaces $L_p(0,1)$ have roundness $p$ for $p \in [1,2]$ [9]. For a geodesic metric space, roundness $r = 2$ is equivalent to being $\mathsf{CAT}(0)$, see [2, Remark 7]. A metric space is called *ultrametric* if the triangle inequality can be strengthened to $\overline{y,z} \leqslant \max(\overline{y,p},\overline{z,p})$ for all points $y,z,p$. Every ultrametric space can be isometrically embedded in a Hilbert space, see, e.g., [11, Corollary 5.4].

A metric space is ultrametric if and only if $r = \infty$, [11, Theorem 5.1]. Then $R = (0,\infty)$, [11, Proposition 2.7]. In general, $R$ is not necessarily an interval [10, Remark p. 254]. But if $(\mathscr{Q},d)$ is a (subset of a) Banach space with the metric $d$ induced by its norm, then $R = (0,r]$ with $r \in [1,2]$, [10, Proposition 4.1.2]. In particular, (16) holds for $\alpha \in (0,2]$ in all inner product spaces. A metric space is called *additive* if

$$\overline{y,q} + \overline{z,p} \leqslant \max(\overline{y,p} + \overline{z,q}, \overline{q,p} + \overline{y,z}) \tag{17}$$

for all points $y,z,q,p$. Every ultrametric space is additive. Every additive metric space is Ptolemaic. Additive metric spaces have roundness $r \geqslant 2$ [11, Proposition 4.1].

### 1.3.4. Martingale theory

Nondecreasing, convex functions with concave derivative play an important role in the Topchii–Vatutin inequality of martingales, see [24, Theorem 2] and [1]: For a suitably integrable martingale $(M_n)_{n\in\mathbb{N}_0}$, we have

$$\mathbb{E}[\tau(|M_n|) - \tau(|M_0|)] \leqslant 2\sum_{k=1}^{n} \mathbb{E}[\tau(|M_k - M_{k-1}|)] \tag{18}$$

for all $\tau \in \mathscr{S}_0$, where $\mathbb{E}[\cdot]$ denotes the expectation. In this context, the functions $\tau \in \mathscr{S}_0$ are named *weakly convex*. Moreover, [24, Lemma 6] gives a weaker version of Theorem 3: Let $\tau \in \mathscr{S}_0$. For $a,b \in [0,\infty)$ with $a \geqslant b$, it was shown that $\tau(a+b) + \tau(a-b) \leqslant 2\tau(a) + 2\tau(b)$.

### 1.3.5. Statistics

Theorem 1 can be applied to prove rates of convergence for certain kinds of means [19]: We may want to calculate a mean value of some sample points in a metric space. One candidate for this is the *Fréchet mean* [14], also called *barycenter*. It is the (set of) minimizer(s) of the squared distance to the sample points. If $Y$ is a random variable with values in a metric space $(\mathscr{Q},d)$, the Fréchet mean is $\arg\min_{q\in\mathscr{Q}} \mathbb{E}[\overline{Y,q}^2]$, where we assume $\mathbb{E}[\overline{Y,q}^2] < \infty$ for all $q \in \mathscr{Q}$. Similarly, one can define the Fréchet median [12] as $\arg\min_{q\in\mathscr{Q}} \mathbb{E}[\overline{Y,q}]$, or a more general $\tau$-Fréchet mean [20] as $\arg\min_{q\in\mathscr{Q}} \mathbb{E}[\tau(\overline{Y,q})]$ for functions $\tau\colon [0,\infty) \to \mathbb{R}$. Given a sequence of independent random variables $Y_1, Y_2, \ldots$ with the same distribution as $Y$, a standard task in statistics is to bound the distance between the sample statistics and its corresponding population version. In our case, assume the $\tau$-Fréchet mean is unique and define

$$m := \arg\min_{q\in\mathscr{Q}} \mathbb{E}[\tau(\overline{Y,q})], \qquad \hat{m}_n := \arg\min_{q\in\mathscr{Q}} \frac{1}{n}\sum_{i=1}^{n} \tau(\overline{Y_i,q}).$$

We want to bound $\overline{\hat{m}_n, m}$ depending on $n$. One can employ quadruple inequalities such as (3) to obtain a suitable upper bound [19, Theorem 1]. This approach is particularly useful, if we do not want to make the assumption that the diameter of the metric space $\sup_{q,p\in\mathscr{Q}} \overline{q,p}$ is finite. With Theorem 1, one can obtain such a bound for $\tau$-Fréchet means with $\tau \in \mathcal{S}$ (under some conditions). We emphasize that this is only possible with (3) and not with (8). Noteworthy examples of $\tau \in \mathcal{S}$ in this context, aside from $\tau = (x \mapsto x^\alpha)$ with $\alpha \in [1,2]$, are the Huber loss $\tau_{\mathsf{H},\delta}$ [16] and the Pseudo-Huber loss $\tau_{\mathsf{pH},\delta}$ [6] for $\delta \in (0,\infty)$,

$$\tau_{\mathsf{H},\delta}(x) := \begin{cases} \frac{1}{2}x^2 & \text{for } x \leqslant \delta\,, \\ \delta(x - \frac{1}{2}\delta) & \text{for } x > \delta\,, \end{cases} \qquad \tau_{\mathsf{pH},\delta}(x) := \delta^2 \left( \sqrt{1 + \frac{x^2}{\delta^2}} - 1 \right),$$

as well as $x \mapsto \ln(\cosh(x))$ [15]. These functions are of great interest in robust statistics and image processing as their respective minimizers combine properties of the classical mean ( $\tau_2$ -Fréchet mean) and the median ( $\tau_1$ -Fréchet mean).

### 1.4. Outline

In the remaining sections, we first discuss the set $\mathcal{S}$ in section 2. The basic ideas for the proof of Theorem 1 and further generalizations of this result are presented in section 3. More technical details can be found in appendix A and even more detail in the arXiv version of this article [21]. We prove Theorem 2 in section 4 and Theorem 3 in section 5. In section 6, we discuss implications of the main results.

## 2. Nondecreasing, convex functions with concave derivative

As the set of nondecreasing, convex functions with concave derivative is central to this article, we start by discussing some basic properties of these functions in this section.

We define $\mathcal{S}$ to be the set of nondecreasing convex functions $\tau\colon [0,\infty) \to \mathbb{R}$ that are differentiable on $(0,\infty)$ with concave derivative $\tau'$. We extend the domain of $\tau'$ to $[0,\infty)$ by setting $\tau'(0) := \lim_{x \searrow 0} \tau'(x)$, which exists as $\tau'$ is nonnegative and nondecreasing. Let $I \subset \mathbb{R}$ be convex. Let $t_0 \in I$ with $t_0 < \sup I$. For a function $f\colon I \to \mathbb{R}$ the *right derivative* at $t_0$ is defined as

$$\partial_+ f(t_0) = \lim_{t \searrow t} \frac{f(t) - f(t_0)}{t - t_0} \tag{19}$$

if the limit exits. For $\tau \in \mathcal{S}$, one can show that the right derivative at 0 exits and $\partial_+ \tau(0) = \tau'(0)$.

Let $\mathcal{C}^k([0,\infty))$ denote the space of $k$-times continuously differentiable functions $[0,\infty) \to \mathbb{R}$, where derivatives at 0 are taken as right derivatives. Denote $\mathcal{S}_0 := \{\tau \in \mathcal{S}\colon \tau(0) = 0\}$. Denote $\mathcal{S}^k := \mathcal{S} \cap \mathcal{C}^k([0,\infty))$ and $\mathcal{S}_0^k := \mathcal{S}_0 \cap \mathcal{C}^k([0,\infty))$ for $k \in \mathbb{N} \cup \{\infty\}$.

The following lemma illustrates that all functions $\tau \in \mathcal{S}^3$ are between a nondecreasing linear function and a parabola that opens upward. It can be shown via Taylor approximations.

LEMMA 1. (Polynomial bounds) *Let* $\tau \in \mathscr{S}^3$. *Let* $x, y \in [0, \infty)$. *Then*

*(i)*

$$\tau(x) + y\tau'(x) \leqslant \tau(x+y) \leqslant \tau(x) + y\tau'(x) + \frac{1}{2}y^2\tau''(x),$$

*(ii)*

$$\tau'(x) \leqslant \tau'(x+y) \leqslant \tau'(x) + y\tau''(x).$$

In the next lemma, we provide useful bounds for the proof of the main results.

LEMMA 2. (Difference bound) *Let* $\tau \in \mathscr{S}$.

*(i)* *Let* $x, y \in [0, \infty)$. *Assume* $x \geqslant y$. *Then*

$$\frac{x-y}{2}\left(\tau'(x) + \tau'(y)\right) \leqslant \tau(x) - \tau(y) \leqslant (x-y)\tau'\left(\frac{x+y}{2}\right).$$

*(ii)* *Let* $x, y \in [0, \infty)$. *Then*

$$\tau(x+y) - \tau(|x-y|) \leqslant 2\min(x,y)\tau'(\max(x,y)).$$

*Proof of Lemma* 2.

(i) For the lower bound, as $\tau'$ is concave,

$$\begin{aligned}
\tau(x) - \tau(y) &= \int_y^x \tau'(u)\mathrm{d}u \\
&\geqslant (x-y)\int_0^1 (1-t)\tau'(y) + t\tau'(x)\,\mathrm{d}t \\
&= \frac{x-y}{2}\left(\tau'(x) + \tau'(y)\right).
\end{aligned}$$

For the upper bound, concavity of $\tau'$ implies the existence of an affine linear function $h$ with $h(u) \geqslant \tau'(u)$ for all $u \in [0, \infty)$ and

$$h\left(\frac{x+y}{2}\right) = \tau'\left(\frac{x+y}{2}\right). \tag{20}$$

Thus,

$$\begin{aligned}
\tau(x) - \tau(y) &\leqslant \int_y^x h(u)\mathrm{d}u \\
&= \frac{x-y}{2}\left(h(x) + h(y)\right) \\
&= (x-y)h\left(\frac{x+y}{2}\right).
\end{aligned}$$

(ii) Follows directly from the upper bound in (i). □

The following lemma is a consequence of $\tau'$ being concave.

LEMMA 3. (Concave derivative) *Let* $\tau \in \mathscr{S}$.

*(i) Let* $x, y \in [0, \infty)$. *Then*

$$\tau'(x+y) \leqslant \tau'(x) + \tau'(y) \leqslant 2\tau'\left(\frac{x+y}{2}\right).$$

*(ii) Let* $a, x \in [0, \infty)$. *Then*

$$\begin{aligned} \tau'(ax) &\geqslant a\tau'(x) \ \textit{for} \ a \leqslant 1, \\ \tau'(ax) &\leqslant a\tau'(x) \ \textit{for} \ a \geqslant 1. \end{aligned}$$

*(iii) Let* $x, y \in [0, \infty)$. *Assume* $y \geqslant x$. *Then*

$$x\tau'(y) \leqslant y\tau'(x).$$

In the proof of Theorem 1, we will first show the result for $\tau \in \mathscr{S}^3$ and then approximate the remaining functions in $\mathscr{S}$ via smooth functions. The following lemma shows that this is possible.

LEMMA 4. (Smooth approximation) *Let* $\tau \in \mathscr{S}$. *Then there is a sequence* $(\tau_n)_{n\in\mathbb{N}} \subset \mathscr{S}^\infty$ *such that* $\tau(x) = \lim_{n\to\infty} \tau_n(x)$ *and* $\tau'(x) = \lim_{n\to\infty} \tau_n'(x)$.

*Proof.* We will smooth $\tau'$ by convolution with a mollifier. The convolution is executed in the group of positive real numbers under multiplication endowed with its Haar measure $\mu(A) = \int_A \frac{1}{x} \mathrm{d}x$ for $A \subset (0, \infty)$ measurable.

For $n \in \mathbb{N}$, let $\varphi_n \in \mathscr{C}^\infty((0,\infty))$ be a sequence of nonnegative functions with support in $[\exp(-1/n), \exp(1/n)]$ and

$$\int_0^\infty \frac{\varphi_n(x)}{x} \mathrm{d}x = 1. \tag{21}$$

Let $\tau \in \mathscr{S}$ with derivative $\tau'$. For $n \in \mathbb{N}$, $x \in [0, \infty)$, we define

$$\tau_n(x) := \tau(0) + \int_0^x \int_0^\infty \frac{\varphi_n(z)}{z} \tau'\left(\frac{y}{z}\right) \mathrm{d}z \mathrm{d}y. \tag{22}$$

Then, for $y \in [0, \infty)$,

$$\tau_n'(y) = \int_0^\infty \frac{\varphi_n(z)}{z} \tau'\left(\frac{y}{z}\right) \mathrm{d}z = \int_{\mathbb{R}} \varphi_n(\mathrm{e}^t) \tau'\left(\mathrm{e}^{\log(y)-t}\right) \mathrm{d}t. \tag{23}$$

Thus, $s \mapsto \tau_n'(\mathrm{e}^s)$ is the convolution of $t \mapsto \varphi_n(\mathrm{e}^t)$ with $t \mapsto \tau'(\mathrm{e}^t)$. Using standard results on convolutions, the mollified function has following properties:

(i) $\tau_n'$ is infinitely differentiable on $(0,\infty)$, because $\varphi_n$ is,

(ii) $\tau_n'$ is nonnegative, nondecreasing, and concave, because $\tau'$ is and $\varphi_n$ is nonnegative,

(iii) $\lim_{n\to\infty} \tau_n'(x) = \tau'(x)$ because $\tau'$ is continuous.

Furthermore, $\tau_n$ is convex, as $\tau_n'$ is nondecreasing and $\lim_{n\to\infty} \tau_n(x) = \tau(x)$ by dominated convergence. Thus, $(\tau_n)_{n\in\mathbb{N}} \subset \mathscr{S}^\infty$, and the sequence has the desired point-wise limits. □

## 3. Outline of the proof of Theorem 1

In the first step of the proof of Theorem 1, we represent general 4-point metric spaces with 6 real-valued parameters. We refer to this representation as a *parametrization*. It converts our problem from the domain of metric geometry to the domain of real analysis. The rest of the proof consists of a complex sequence of elementary calculus arguments. This sequence may seem difficult to discover. To aid this process, an extensive application of computer-assisted numerical assessments was employed. The inequality (3) and transformations of it were evaluated on a grid of the parameter space and for a finite set of functions $\tau$. This computational tool played a crucial role in guiding the proof. It helped to identify steps that would not be useful and indicated steps with potential merit.

### 3.1. Parametrization

For the proof of Theorem 1, we use a 4-point parametrization, see Figure 2. It is based on repeated application of the Euclidean cosine formula. Its parameter space is a subset of $[0,\infty)^3 \times [-1,1]^3$ with rather complex constraints. We then relax the problem by dropping these constraints.

PROPOSITION 2. (4-point parametrization)

*(i) Let $(\mathcal{Q},d)$ be a metric space and $y,z,q,p \in \mathcal{Q}$. Set $a := \overline{z,p}$, $c := \overline{y,p}$, $b := \overline{q,p}$, $s := \cos(\measuredangle ypq)$, $r := \cos(\measuredangle zpq)$, $t := \cos(\measuredangle ypz)$, with all angles measured as in an isometric 3-point embedding in the Euclidean plane. Then,*

$$\overline{y,z}^2 = a^2 + c^2 - 2tac\,, \quad \overline{y,q}^2 = c^2 + b^2 - 2scb\,, \quad \overline{z,q}^2 = a^2 + b^2 - 2rab\,.$$

*Furthermore, $a,b,c \in [0,\infty)$, $r,s,t \in [-1,1]$, and*

$$\begin{aligned} -tac &\leqslant b^2 - rab - scb + \sqrt{a^2 - 2rab + b^2}\sqrt{c^2 - 2scb + b^2}\,, \\ -rab &\leqslant c^2 - tac - scb + \sqrt{a^2 - 2tac + c^2}\sqrt{c^2 - 2scb + b^2}\,, \\ -scb &\leqslant a^2 - rab - tac + \sqrt{a^2 - 2rab + b^2}\sqrt{a^2 - 2tac + c^2}\,. \end{aligned} \tag{24}$$

(ii) *For all* $a,b,c \in [0,\infty)$, $r,s,t \in [-1,1]$ *that fulfill* (24)*, there is a metric space* $(\mathcal{Q},d)$ *with a quadruple of points* $y,z,q,p \in \mathcal{Q}$ *such that*

$$a = \overline{z,p}\,, \qquad c = \overline{y,p}\,, \qquad b = \overline{q,p}\,,$$
$$\overline{y,z}^2 = a^2 + c^2 - 2tac\,, \quad \overline{y,q}^2 = c^2 + b^2 - 2scb\,, \quad \overline{z,q}^2 = a^2 + b^2 - 2rab\,.$$

*Proof.*

(i) The inequalities (24) are due to the triangle inequality,

$$\overline{y,z} \leqslant \overline{y,q} + \overline{z,q}\,, \qquad \overline{z,q} \leqslant \overline{y,z} + \overline{y,q}\,, \qquad \overline{y,q} \leqslant \overline{y,z} + \overline{z,q}\,.$$

(ii) Define a four point set $\mathcal{Q}$ with elements named $y,z,q,p$. Define $d\colon \mathcal{Q}\times\mathcal{Q} \to [0,\infty)$ with the equations given in the lemma, extended by symmetry and $d(x,x) = 0$ for all $x \in \mathcal{Q}$. By construction, $d$ is a semimetric [8] (vanishing distance for non-identical points allowed) so that identifying points with distance 0 yields a metric space. □

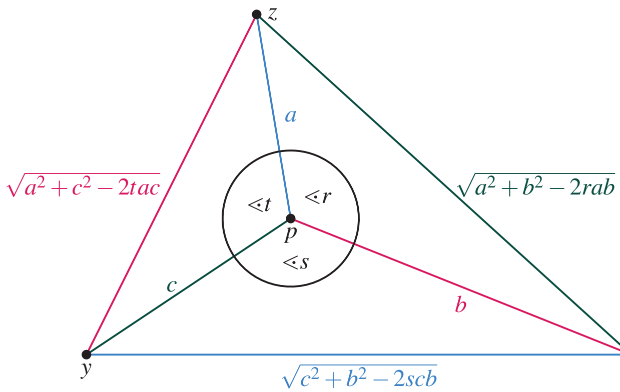


Figure 2: *A 4-point parametrization. We denote* $\sphericalangle x := \arccos(x)$.

With this parametrization and

$$a^2 - c^2 - (a^2 - 2rab + b^2) + (c^2 - 2scb + b^2) = 2b\,(ra - sc)\,, \tag{25}$$

(5) can be expressed as

$$b\,(ra - sc) \leqslant b\sqrt{a^2 + c^2 - 2tac}\,, \tag{26}$$

and (3) becomes

$$\tau(a) - \tau(c) - \tau_{\surd}\big(a^2 - 2rab + b^2\big) + \tau_{\surd}\big(c^2 - 2scb + b^2\big) \leqslant L_\tau\, b\, \tau'(a^2 + c^2 - 2tac)\,, \tag{27}$$

where we use the shorthand $\tau_{\surd}(x) := \tau(\sqrt{x})$. Thus, Theorem 1 is equivalent to showing that (26) implies (27) for all $a,b,c \in [0,\infty)$, $r,s,t \in [-1,1]$ that fulfill (24). We will prove a stronger but simpler looking result in section A of the appendix:

THEOREM 4. *Let $a,b,c \geqslant 0$, $r,s \in [-1,1]$, and $\tau \in \mathscr{S}_0^3$. Then*

$$\tau(a) - \tau(c) - \tau_{\surd}\left(a^2 - 2rab + b^2\right) + \tau_{\surd}\left(c^2 - 2scb + b^2\right) \leqslant 2b\tau'(\max(ra - sc, |a-c|)) . \tag{28}$$

### 3.2. Remaining proof steps

From Theorem 4, we obtain a slightly stronger result than Theorem 1 by relaxing (5):

THEOREM 5. *Let $\tau \in \mathscr{S}$. Let $(\mathscr{Q}, d)$ be a metric space. Let $y,z,q,p \in \mathscr{Q}$. Assume, there is $L \in [2,\infty)$ such that*

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leqslant L\,\overline{q,p}\,\overline{y,z} . \tag{29}$$

*Then*

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \leqslant L\,\overline{q,p}\,\tau'(\overline{y,z}) . \tag{30}$$

*Proof that Theorem 4 implies Theorem 5.* Let $\tau \in \mathscr{S}_0^3$. Using (25), the metric version of (28) is

$$\begin{aligned}&\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \\ &\leqslant 2\,\overline{q,p}\,\tau'\left(\max\left(\frac{\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2}{2\,\overline{q,p}}, |\overline{z,p} - \overline{y,p}|\right)\right) .\end{aligned} \tag{31}$$

Using (29) and the triangle inequality, we bound the right-hand side of (31), by

$$2\,\overline{q,p}\,\tau'\left(\max\left(\frac{L\,\overline{y,z}}{2}, \overline{y,z}\right)\right) . \tag{32}$$

With Lemma 3 (ii) and $L/2 \geqslant 1$, we obtain (30).

We extend the result now shown for $\tau \in \mathscr{S}_0^3$ to $\tau \in \mathscr{S}$: We use Lemma 4 to construct a smooth sequence $(\tau_n) \subset \mathscr{S}_0^3$ that approximates $\tau \in \mathscr{S}$. One can easily show that (30) also holds in the limit. Furthermore, the condition $\tau(0) = 0$ can be dropped, as we can add a constant to $\tau$ without changing the validity of (30). □

Theorem 1 follows from Theorem 5 by fixing $L = 2$. The remaining part of the main proof, i.e., the proof of Theorem 4, is given in the appendix section A. Figure 3 gives an overview of how the different intermediate results presented above and below relate to each other.

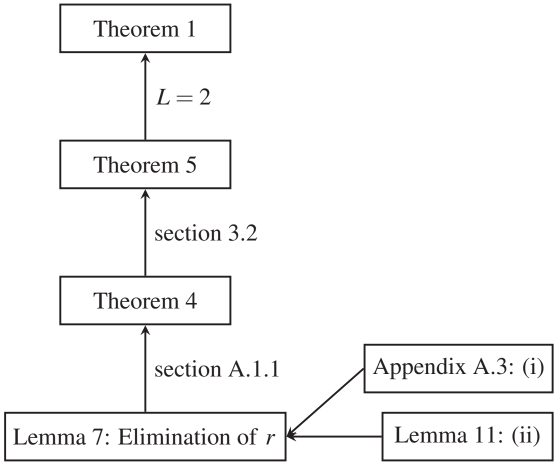


Figure 3: *Overview of theorems and lemmas in the proof of Theorem 1.*

## 4. Proof of Theorem 2

We only prove $L_\tau \geqslant 1$ for $\tau \in \mathscr{S}$ as the upper bound follows from Theorem 1.

Let $u \in (0,\infty)$ be such that $\tau'(u) \neq 0$. Then $\tau'(u) > 0$. Let $\varepsilon \in (0,u)$. In the metric space of the real line with Euclidean metric, we choose $z = q = 0$, $p = \varepsilon$, $y = u$, see Figure 4. Then (3) becomes

$$\tau(u) - \tau(u-\varepsilon) - \tau(0) + \tau(\varepsilon) \leqslant L_\tau \, \varepsilon \, \tau'(u)\,.$$

As $\tau$ is nondecreasing, we have $\tau(0) = \lim_{x \searrow 0} \tau(x) \leqslant \tau(\varepsilon)$. Thus,

$$L_\tau \tau'(u) \geqslant \frac{\tau(u) - \tau(u-\varepsilon)}{\varepsilon} \xrightarrow{\varepsilon \to 0} \tau'(u)\,.$$

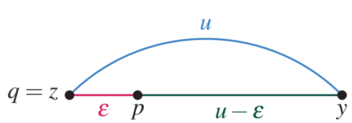


Figure 4: *Construction for the proof of Theorem 2.*

## 5. Proof of Theorem 3

The proof of Theorem 3 is inspired by the proofs of [10, Proposition 4.1.1, Proposition 4.1.2]. We first state two useful lemmas. This first one is a well-known property of concave functions.

LEMMA 5. *Let* $f\colon [0,\infty) \to \mathbb{R}$. *Assume* $f$ *is concave. Let* $a,b \in [0,\infty)$ *with* $a \geqslant b$. *Then* $x \mapsto f(a+x)+f(b-x)$ *is nonincreasing. If additionally* $f(0) \geqslant 0$, *then* $f$ *is subadditive.*

LEMMA 6. *Let* $f\colon [0,\infty) \to \mathbb{R}$. *Assume* $f(0) \geqslant 0$, $f$ *is nondecreasing, and* $f$ *is concave. Let* $x_1,\dots,x_6 \in [0,\infty)$. *Assume* $\max(x_1,x_2,x_3,x_4) \leqslant \max(x_5,x_6)$ *and* $x_1+x_2+x_3+x_4 \geqslant x_5+x_6$. *Then*

$$f(x_1)+f(x_2)+f(x_3)+f(x_4) \geqslant f(x_5)+f(x_6)\,. \tag{33}$$

*Proof.* Without loss of generality, assume $x_1 \geqslant x_2 \geqslant x_3 \geqslant x_4$ and $x_5 \geqslant x_6$.

First consider the case $x_1 \geqslant x_6$. We decrease $x_5$ and increase $x_6$ while holding $x_5+x_6$ constant until one $x_\bullet$ on the right-hand side coincides with one $x_\bullet$ one the left-hand side. By Lemma 5, this can only increase the right-hand side of (33). If $\{x_1,x_2,x_3,x_4\} \cap \{x_5,x_6\} \neq \emptyset$, we can subtract the term with the value in the intersection from (33). The inequality of the form $f(x_1)+f(x_2)+f(x_3) \geqslant f(x_1+x_2+x_3) \geqslant f(x_5)$ for $x_5 \geqslant x_1+x_2+x_3$ is obtained using subadditivity of $f$, see Lemma 5, and the assumption that $f$ is nondecreasing.

Now consider the case $x_1 < x_6$. Set $s := (x_5+x_6)/2$. Using Lemma 5, we obtain $f(x_5)+f(x_6) \leqslant 2f(s)$. Furthermore $x_1 \leqslant s$ and $x_1+x_2+x_3+x_4 \leqslant 2s$. Thus, again using Lemma 5 and the assumption that $f$ is nondecreasing, we can increase $x_1$ and $x_2$ while decreasing $x_3$ and $x_4$ to 0 to get

$$f(x_1)+f(x_2)+f(x_3)+f(x_4) \geqslant 2f(s)+2f(0)\,. \tag{34}$$

As $f(0) \geqslant 0$, we arrive at the desired result. □

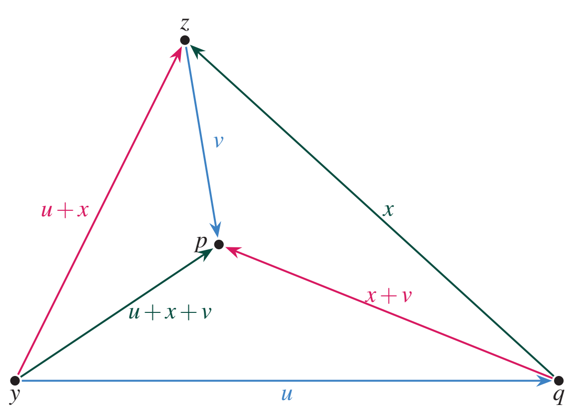


Figure 5: *Four points in a vector space* $V$. *Their relative position is described by three vectors* $u,v,x \in V$.

*Proof of Theorem* 3. For any four points $y,z,q,p \in V$, we have

$$\|y-q\|^2 - \|y-p\|^2 - \|z-q\|^2 + \|z-p\|^2 = 2\langle y-z, p-q\rangle \leqslant \|q-p\|^2 + \|y-z\|^2 . \tag{35}$$

Let $u,v \in V$. Consider a parallelogram with vertices $(0,(u-v)/2,u,(u+v)/2)$. It has the diagonals $u$ and $v$ and the largest diagonal is not smaller than the largest side. As $\tau \in \mathscr{S}_0$, $\tau_{\sqrt{}}$ is nonnegative, nondecreasing, and concave. Thus, we can apply Lemma 6 to

$$x_1 = x_2 = \left\|\frac{u-v}{2}\right\|^2, \quad x_3 = x_4 = \left\|\frac{u+v}{2}\right\|^2, \quad x_5 = \|u\|^2, \quad x_6 = \|v\|^2, \tag{36}$$

where $x_1 + x_2 + x_3 + x_4 \geqslant x_5 + x_6$ is ensured by (35). We obtain

$$\tau(\|u\|) + \tau(\|v\|) \leqslant 2\tau\left(\left\|\frac{u-v}{2}\right\|\right) + 2\tau\left(\left\|\frac{u+v}{2}\right\|\right) . \tag{37}$$

To extend the result from parallelograms to any quadrilateral, we note that $\tau$ is nondecreasing and convex, and apply Lemma 5: For every $x \in V$,

$$2\tau\left(\left\|\frac{u-v}{2}\right\|\right) \leqslant \tau(\|x\|) + \tau(\|u-v-x\|) , \tag{38}$$

$$2\tau\left(\left\|\frac{u+v}{2}\right\|\right) \leqslant \tau(\|u-x\|) + \tau(\|v+x\|) . \tag{39}$$

By appropriate choice of $u,v,x$ for a given quadrilateral with vertices $y,z,q,p$, see Figure 5, we have shown

$$\tau(\|y-q\|) + \tau(\|z-p\|) \leqslant \tau(\|q-p\|) + \tau(\|y-z\|) + \tau(\|y-p\|) + \tau(\|z-q\|) \tag{40}$$

and finished the proof of Theorem 3. $\square$

## 6. Corollaries

With Theorem 1 and Theorem 3, we have shown new fundamental inequalities in metric spaces and inner product spaces that are related to the Cauchy–Schwarz and the Triangle inequalities. In this section, we discuss some corollaries of these results.

### 6.1. Symmetries

Figure 6 illustrates the symmetries in quadruple inequalities. Sides of the same color contribute essentially in the same way in the inequality. In (3), the diagonals of Figure 6 come up in non-exchangeable terms. But they can be swapped in the assumption (5). Thus, if the conditions of Theorem 1 are fulfilled, we have, for $\tau \in \mathscr{S}$,

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \leqslant 2\min(\overline{q,p}\,\tau'(\overline{y,z}), \overline{y,z}\,\tau'(\overline{q,p})) . \tag{41}$$

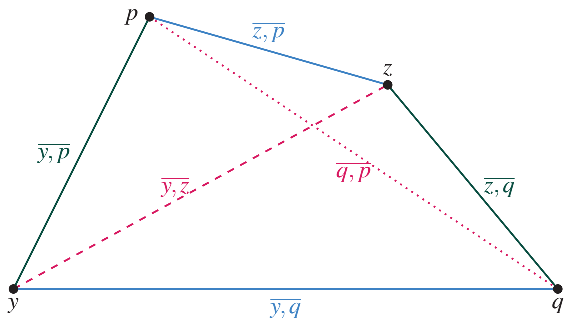


Figure 6: *Four points as a quadrilateral. The sides of the quadrilateral show up on the left of the quadruple inequalities (the terms of opposite sides have the same sign); the diagonals form the right-hand side.*

Furthermore, swapping $y$ and $z$ or $q$ and $p$ does not influence the right-hand side but changes the sign on the left-hand side. Thus, assuming

$$\left|\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2\right| \leqslant 2\,\overline{q,p}\,\overline{y,z} \tag{42}$$

we get

$$|\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p})| \leqslant 2\min(\overline{q,p}\,\tau'(\overline{y,z}), \overline{y,z}\,\tau'(\overline{q,p}))\,. \tag{43}$$

Further bounds of the right-hand side are shown in the next subsection.

### 6.2. Bounds for the right-hand side

COROLLARY 1. *Let* $\tau \in \mathscr{S}_0$. *Let* $(\mathscr{Q}, d)$ *be a metric space. Let* $y,z,q,p \in \mathscr{Q}$. *Assume*

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leqslant 2\,\overline{q,p}\,\overline{y,z}\,. \tag{44}$$

*Then the value*

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \tag{45}$$

*is bounded from above by all of the following values:*

*(i)* $2\min(\overline{q,p}\,\overline{y,z})\,\tau'(\max(\overline{q,p}\,\overline{y,z}))$,

*(ii)* $2\,\overline{q,p}^{\beta}\,\overline{y,z}^{1-\beta}\,\tau'\left(\overline{q,p}^{1-\beta}\,\overline{y,z}^{\beta}\right)$ *for all* $\beta \in [0,1]$,

*(iii)* $2\,(\beta\,\overline{q,p} + (1-\beta)\overline{y,z})\,\tau'((1-\beta)\overline{q,p} + \beta\,\overline{y,z})$ *for all* $\beta \in [0,1]$,

*(iv)* $2\sqrt{\overline{q,p}\,\overline{y,z}}\,\tau'(\sqrt{\overline{q,p}\,\overline{y,z}})$,

*(v)* $(\overline{q,p} + \overline{y,z})\,\tau'\left(\frac{\overline{q,p}+\overline{y,z}}{2}\right)$,

*(vi)* $4\tau(\sqrt{\overline{q,p}\,\overline{y,z}})$,

*(vii)* $4\tau\left(\frac{\overline{q,p}+\overline{y,z}}{2}\right)$,

*(viii)* $2\tau(\overline{q,p})+2\tau(\overline{y,z})$.

*Proof.* We first apply Theorem 1 twice, to $(y,z,q,p)$ and to $(q,p,y,z)$, to obtain

$$\tau(\overline{y,q})-\tau(\overline{y,p})-\tau(\overline{z,q})+\tau(\overline{z,p})\leqslant 2\min(\overline{q,p}\,\overline{y,z})\tau'(\max(\overline{q,p}\,\overline{y,z}))\,. \tag{46}$$

This shows (i). Let $a,b\in[0,\infty)$ and $\beta\in[0,1]$. Next we use Lemma 3 (ii) and the weighted arithmetic–geometric mean inequality,

$$\begin{aligned}\min(a,b)\tau'(\max(a,b)) &\leqslant a^{\beta}b^{1-\beta}\tau'(a^{1-\beta}b^{\beta})\\ &\leqslant(\beta a+(1-\beta)b)\,\tau'((1-\beta)a+\beta b)\,.\end{aligned}$$

Applying these inequalities to (46) shows (ii) and (iii), and their special cases (iv) and (v), where $\beta=1/2$. By Lemma 2 (i) with $y=0$, we have

$$x\tau'(x)\leqslant 2\tau(x) \tag{47}$$

for all $x\in[0,\infty)$. Thus,

$$\begin{aligned}\min(a,b)\tau'(\max(a,b)) &\leqslant \sqrt{ab}\,\tau'(\sqrt{ab})\\ &\leqslant 2\tau(\sqrt{ab})\,,\end{aligned}$$

which yields (vi). The remaining parts (vii) and (viii), can be obtained using (47) and Jensen's inequality:

$$\begin{aligned}\min(a,b)\tau'(\max(a,b)) &\leqslant \frac{a+b}{2}\tau'\left(\frac{a+b}{2}\right)\\ &\leqslant 2\tau\left(\frac{a+b}{2}\right)\\ &\leqslant \tau(a)+\tau(b)\,. \quad \square\end{aligned}$$

### 6.3. Corollaries for special cases

We apply Theorem 1, Theorem 3, and Proposition 1 for a triple of points (a quadruple of points with two identical points) and for parallelograms in inner product spaces to demonstrate the main results.

COROLLARY 2. (For three points) *Let $(\mathscr{Q},d)$ be a metric space. Let $y,q,p\in\mathscr{Q}$.*

*(i) Let $\tau\in\mathscr{S}_0$. Then*

$$\tau(\overline{y,q})-\tau(\overline{y,p})+\tau(\overline{q,p})\leqslant 2\overline{q,p}\,\tau'(\overline{y,q})\,. \tag{48}$$

*(ii) Let $\alpha\in[1,2]$. Then*

$$\overline{y,q}^{\alpha}-\overline{y,p}^{\alpha}+\overline{q,p}^{\alpha}\leqslant \alpha 2^{2-\alpha}\,\overline{q,p}\,\overline{y,q}^{\alpha-1}\,. \tag{49}$$

*Proof.* By the triangle inequality, $|\overline{y,q}-\overline{q,p}| \leqslant \overline{y,p}$. After squaring this inequality, we obtain (5) with $z=q$, which is

$$\overline{y,q}^2-\overline{y,p}^2+\overline{q,p}^2 \leqslant 2\overline{q,p}\,\overline{y,q}. \tag{50}$$

Thus, Theorem 1 implies (48) and Proposition 1 implies (49). □

COROLLARY 3. (In inner product spaces) *Let* $(V,\langle\cdot,\cdot\rangle)$ *be an inner product space with induced norm* $\|\cdot\|$. *Let* $u,v\in V$.

*(i) Let* $\tau\in\mathscr{S}_0$. *Then*

$$\tau(\|u\|)-\tau(\|v\|) \leqslant \|u-v\|\tau'(\|u+v\|). \tag{51}$$

*(ii) Let* $\alpha\in[1,2]$. *Then*

$$\|u\|^{\alpha}-\|v\|^{\alpha} \leqslant \alpha 2^{1-\alpha}\|u-v\|\|u+v\|^{\alpha-1}. \tag{52}$$

*(iii) Let* $\tau\in\mathscr{S}_0$. *Then*

$$\tau(\|u+v\|)+\tau(\|u-v\|) \leqslant 2\tau(\|u\|)+2\tau(\|v\|). \tag{53}$$

*(iv) Let* $\alpha\in[1,2]$. *Then*

$$\|u+v\|^{\alpha}+\|u-v\|^{\alpha} \leqslant 2\|u\|^{\alpha}+2\|v\|^{\alpha}. \tag{54}$$

*Proof.* For the first two parts, set $y=0$, $z=u+v$, $q=u$, $p=v$; for the last two parts, set $y=0$, $z=v$, $q=u+v$, $p=u$. Then apply Theorem 1, Theorem 3, and Proposition 1. □

REMARK 1. Recall the *parallelogram law*: Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space with induced norm $\|\cdot\|$. Let $u,v\in V$. Then

$$\|u+v\|^2+\|u-v\|^2 \leqslant 2\|u\|^2+2\|v\|^2, \tag{55}$$

which is also true with equality. Thus, we can say that Corollary 3 generalizes the parallelogram law.

## A. Proof of Theorem 4

This section gives a more detailed view of the proof of Theorem 4. The proof contains many simple calculus arguments, which do not seem to be particularly interesting beyond the scope of this proof. Thus, not every step is laid out in full detail here. An extended version of the proof with a step-by-step description is given in [21].

Before we start with the main proof, let us remark on some technicality: If we want to show $f(x)\leqslant 0$ for all $x\in A$, where $A\subset\mathbb{R}^n$, for a continuous function $f\colon A\to\mathbb{R}$,

| $f$ | $\tau$ | $\tau'$ | $\tau''$ | $\tau'''$ | $\tau_{\surd}$ | $\tau'_{\surd}$ | $\tau''_{\surd}$ | $\tau'''_{\surd}$ |
|---|---|---|---|---|---|---|---|---|
| $f(0)$ | 0 | $\geqslant 0$ | $\geqslant 0$ | $\leqslant 0$ | 0 | $\geqslant 0$ | $\leqslant 0$ | |
| $f(x)$ | $\geqslant 0$ | $\geqslant 0$ | $\geqslant 0$ | $\leqslant 0$ | $\geqslant 0$ | $\geqslant 0$ | $\leqslant 0$ | |
| $f$ monotone | ↗ | ↗ | ↘ | | ↗ | ↘ | | |
| $f$ curvature | ⌣ | ⌢ | | | ⌢ | | | |

Table 1: *Properties of $\tau \in \mathscr{S}_0^3$ and function $\tau_{\surd}(x) = \tau(\sqrt{x})$. We denote ↗ (↘) to indicate a nondecreasing (nonincreasing) function and ⌣ (⌢) for a convex (concave) function.*

it is enough to prove the inequality on a dense subset of $A$. We use this fact in the following. If we write an expression with a quotient, we silently restrict the domains of the real parameters in all statements about this expression to a domain on which the denominator is not 0. The restricted domain, will always be dense in the unrestricted domain.

Recall $\tau_{\surd}(x) = \tau(\sqrt{x})$. Properties of $\tau \in \mathscr{S}_0^3$ and the corresponding $\tau_{\surd}$ are summarized in Table 1 for reference.

### A.1. Lemma 7: elimination of $r$

We will show that the following lemma implies Theorem 4.

LEMMA 7. (Elimination of $r$) *Let $\tau \in \mathscr{S}_0^3$.*

*(i) For all $a,b,c \in [0,\infty)$, $s \in [-1, \min(1, 2\frac{a}{c} - 1)]$, we have*

$$\tau(a) - \tau(c) - \tau(|a-b|) + \tau_{\surd}(c^2 - 2scb + b^2) \leqslant 2b\tau'(a - sc) \ . \tag{56}$$

*(ii) For all $a,b,c \in [0,\infty)$ with $a \geqslant c$, we have*

$$\tau(a) - \tau(c) - \tau_{\surd}((a-b)^2 - 4bc) + \tau_{\surd}(b+c) \leqslant 2b\tau'(a-c)\,. \tag{57}$$

#### A.1.1. Proof that Lemma 7 implies Theorem 4

For this proof, we first show some auxiliary lemmas. We distinguish the cases $ra - sc \leqslant |a-c|$ and $ra - sc \geqslant |a-c|$ as well as $a \geqslant c$ and $c \geqslant a$. Some trivial implications of these cases are recorded in following lemma.

LEMMA 8. *Let $a,b,c \geqslant 0$, $r,s \in [-1,1]$. Then*

$$ra - sc \geqslant a - c \quad \Leftrightarrow \quad s \leqslant (r-1)\frac{a}{c} + 1 \quad \Leftrightarrow \quad r \geqslant (s-1)\frac{c}{a} + 1\,,$$
$$ra - sc \geqslant c - a \quad \Leftrightarrow \quad s \leqslant (r+1)\frac{a}{c} - 1 \quad \Leftrightarrow \quad r \geqslant (s+1)\frac{c}{a} - 1\,.$$

Denote

$$\ell_\tau(a,b,c,r,s) := \tau(a) - \tau(c) - \tau_{\surd}\big(a^2 - 2rab + b^2\big) + \tau_{\surd}\big(c^2 - 2scb + b^2\big)\,,$$
$$F_\tau(a,b,c,r,s) := \ell_\tau(a,b,c,r,s) - 2b\tau'(ra - sc)\,.$$

For $ra - sc \geqslant |a-c|$, we want to show $F_\tau(a,b,c,r,s) \leqslant 0$. Because of the next lemma, which has a straight forward proof, we can reduce the number of values of $r$ for which we need to check this inequality.

LEMMA 9. (Convexity in $r$) *Let* $\tau \in \mathscr{S}_0^3$. *Let* $a,b,c \geqslant 0$, $s,r \in [-1,1]$. *Assume* $ra - sc \geqslant 0$. *Then*

$$\partial_r^2 F_\tau(a,b,c,r,s) \geqslant 0\,.$$

In the case $|a-c| \geqslant ra - sc$, the right-hand side of (27) does not depend on $r$ or $s$. Thus, we will only need to check the inequality with the left-hand side $\ell_\tau$ maximized in $r$ and $s$.

LEMMA 10. (Maximizing the left-hand side for $|a-c| \geqslant ra - sc$) *Let* $\tau \in \mathscr{S}_0^3$. *Let* $a,b,c \in [0,\infty)$, $r,s \in [-1,1]$. *Assume* $|a-c| \geqslant ra - sc$.

*(i) If* $a \geqslant c$ *and* $a^2 \leqslant c^2 + 2ab - 2cb$, *then*

$$\ell_\tau(a,b,c,r,s) \leqslant \tau(a) - \tau(c) - \tau(|a-b|) + \tau(|c-b|)\,.$$

*(ii) If* $a \geqslant c$ *and* $a^2 \geqslant c^2 + 2ab - 2cb$, *then*

$$\ell_\tau(a,b,c,r,s) \leqslant \tau(a) - \tau(c) - \tau_{\surd}((a-b)^2 + 4cb) + \tau(c+b)\,.$$

*(iii) If* $a \leqslant c$, *then*

$$\ell_\tau(a,b,c,r,s) \leqslant \tau(a) - \tau(c) - \tau_{\surd}(|a-b|) + \tau_{\surd}((c+b)^2 - 4ab)\,.$$

*Proof.* As $\tau_{\surd}$ is nondecreasing, $s \mapsto \ell_\tau(a,b,c,r,s)$ is nonincreasing and $r \mapsto \ell_\tau(a,b,c,r,s)$ nondecreasing, i.e., for $s_0, r_0 \in [-1,1]$,

$$\max_{s \geqslant s_0, r \leqslant r_0} \ell_\tau(a,b,c,r,s) = \ell_\tau(a,b,c,r_0,s_0)\,.$$

*Case* 1: $a \geqslant c$. For $r \in [-1,1]$, set $s_{\mathsf{min}}(r) := (r-1)\frac{a}{c} + 1$, cf. Lemma 8. Define

$$\begin{aligned} f(r) &:= \ell_\tau(a,b,c,r,s_{\mathsf{min}}(r)) \\ &= \tau(a) - \tau(c) - \tau_{\surd}(a^2 - 2rab + b^2) + \tau_{\surd}(c^2 - 2rab + 2ab - 2cb + b^2)\,. \end{aligned}$$

Then

$$\frac{f'(r)}{2ab} = \tau_{\surd}'(a^2 - 2rab + b^2) - \tau_{\surd}'(c^2 - 2rab + 2ab - 2cb + b^2)\,.$$

*Case* 1.1: $a^2 \leqslant c^2 + 2ab - 2cb$. As $\tau'_{\surd}$ is nonincreasing, we have

$$a^2 - 2rab + b^2 \leqslant c^2 - 2rab + 2ab - 2cb + b^2,$$
$$\tau'_{\surd}(a^2 - 2rab + b^2) \geqslant \tau'_{\surd}(c^2 - 2rab + 2ab - 2cb + b^2).$$

Thus, $f'(r) \geqslant 0$ and $f$ is maximal at $r = r_{\mathsf{max}} = 1$, with $s_{\mathsf{min}}(r) = 1$. Hence,

$$\ell_\tau(a,b,c,r,s) \leqslant f(1) = \tau(a) - \tau(c) - \tau(|a-b|) + \tau(|c-b|).$$

*Case* 1.2: $a^2 \geqslant c^2 + 2ab - 2cb$. As $\tau'_{\surd}$ is nonincreasing, we have

$$a^2 - 2rab + b^2 \geqslant c^2 - 2rab + 2ab - 2cb + b^2,$$
$$\tau'_{\surd}(a^2 - 2rab + b^2) \leqslant \tau'_{\surd}(c^2 - 2rab + 2ab - 2cb + b^2).$$

Thus, $f'(r) \leqslant 0$ and $f$ is maximal at $r = r_{\mathsf{min}} = 1 - 2\frac{c}{a}$, with $s_{\mathsf{min}}(r) = -1$. Hence,

$$\ell_\tau(a,b,c,r,s) \leqslant f(r_{\mathsf{min}}) = \tau(a) - \tau(c) - \tau_{\surd}((a-b)^2 + 4cb) + \tau(c+b).$$

*Case* 2: $a \leqslant c$. For $r \in [-1,1]$, set $s_{\mathsf{min}}(r) := (r+1)\frac{a}{c} - 1$, cf. Lemma 8. Define

$$\begin{aligned} f(r) &:= \ell_\tau(a,b,c,r,s_{\mathsf{min}}(r)) \\ &= \tau(a) - \tau(c) - \tau_{\surd}(a^2 - 2rab + b^2) + \tau_{\surd}(c^2 - 2rab - 2ab + 2cb + b^2). \end{aligned}$$

Then
$$\frac{f'(r)}{2ab} = \tau'_{\surd}(a^2 - 2rab + b^2) - \tau'_{\surd}(c^2 - 2rab - 2ab + 2cb + b^2).$$

*Case* 2.1: $a^2 \leqslant c^2 - 2ab + 2cb$. As $\tau'_{\surd}$ is nonincreasing, we have

$$a^2 - 2rab + b^2 \leqslant c^2 - 2rab - 2ab + 2cb + b^2,$$
$$\tau'_{\surd}(a^2 - 2rab + b^2) \geqslant \tau'_{\surd}(c^2 - 2rab - 2ab + 2cb + b^2).$$

Thus, $f'(r) \geqslant 0$ and $f$ is maximal at $r = r_{\mathsf{max}} = 1$, with $s_{\mathsf{min}}(r) = 2\frac{a}{c} - 1$. Hence,

$$\ell_\tau(a,b,c,r,s) \leqslant f(1) = \tau(a) - \tau(c) - \tau_{\surd}(|a-b|) + \tau_{\surd}((c+b)^2 - 4ab).$$

*Case* 2.2: $a^2 \geqslant c^2 - 2ab + 2cb$. This cannot happen for $a < c$. Hence, the proof is finished. □

*Proof that Lemma 7 implies Theorem 4.*

*Case* 1: $|a-c| \geqslant ra - sc$.

*Case* 1.1: $a \geqslant c$. We can apply Lemma 10 and it suffices to show

$$\tau(a) - \tau(c) - \tau(|a-b|) + \tau(|c-b|) \leqslant 2b\tau'(a-c) \tag{58}$$

for $a^2 \leqslant c^2 + 2ab - 2cb$, and for $a^2 \geqslant c^2 + 2ab - 2cb$,

$$\tau(a) - \tau(c) - \tau_{\surd}((a-b)^2 + 4cb) + \tau_{\surd}(c+b) \leqslant 2b\tau'(a-c)\,. \tag{59}$$

The latter is exactly Lemma 7 (ii). The former follows from Lemma 7 (i) with $s \in \{-1,1\}$.

*Case* 1.2: $a \leqslant c$. We can apply Lemma 10 and it suffices to show

$$\tau(a) - \tau(c) - \tau_{\surd}(|a-b|) + \tau_{\surd}((c+b)^2 - 4ab) \leqslant 2b\tau'(c-a)\,. \tag{60}$$

This follows from Lemma 7 (i) with $s = 2\frac{a}{c} - 1$.

*Case* 2: $|a-c| \leqslant ra - sc$.

*Case* 2.1: $a \geqslant c$. In this case, $F_\tau(a,b,c,r,s) \leqslant 0$ implies (28). As $r \mapsto F_\tau(a,b,c,r,s)$ is convex by Lemma 9, it suffices to show $F_\tau(a,b,c,r,s) \leqslant 0$ for the extreme values of $r$ in order to establish this inequality for all $r$. By Lemma 8,

$$r \in \left[(s-1)\frac{c}{a} + 1,\, 1\right]. \tag{61}$$

For maximal $r$, by Lemma 7 (i), we have $F_\tau(a,b,c,1,s) \leqslant 0$. For minimal $r$, we have $ra - sc = a - c$. Thus, we are in case 1.1.

*Case* 2.2: $a \leqslant c$. As in case 2.1, it suffices show $F_\tau(a,b,c,r,s) \leqslant 0$ for extreme values of $r$. By Lemma 8,

$$r \in \left[(s+1)\frac{c}{a} - 1,\, 1\right]. \tag{62}$$

For maximal $r$, by Lemma 7 (i), we have $F_\tau(a,b,c,1,s) \leqslant 0$. For minimal $r$, we have $ra - sc = c - a$. Thus, we are in case 1.2. □

### A.2. Proof of Lemma 7 (ii)

LEMMA 11. *Let $\tau \in \mathscr{S}_0^3$. Let $a,b,c \in [0,\infty)$. Assume $a \geqslant c$. Then*

$$\tau(a) - \tau(c) - \tau_{\surd}((a-b)^2 - 4bc) + \tau_{\surd}(b+c) \leqslant 2b\tau'(a-c)\,. \tag{63}$$

*Proof.* Define

$$f(a,b,c) := \tau(a) - \tau(c) - \tau_{\surd}((a-b)^2 + 4cb) + \tau(c+b) - 2b\tau'(a-c)\,. \tag{64}$$

By distinguishing the cases $a-b-2c \geqslant 0$ and $a-b-2c \leqslant 0$, some simple calculations show

$$2(a-b-2c)\tau'_{\surd}((a-b)^2 + 4cb) + \tau'(c+b) - 2\tau'(a-c) \leqslant 0\,. \tag{65}$$

Thus, $\partial_b f(a,b,c) \leqslant 0$. Hence, as $b \geqslant 0$, we have $f(a,b,c) \leqslant f(a,0,c) = 0$. □

### A.3. Outline of proof of Lemma 7 (i)

The proof of Lemma 7 (i) follows a similar pattern as part (ii): Define the function $f(a,b,c,s)$ as left-hand side minus right-hand side of (56). Take suitable partial derivatives (potentially after re-parameterization of the domain of $f$) until one can easily show that the respective derivative is always nonpositive or nonnegative, which allows one to fix one of the parameters at an extreme value of the domain. This procedure is repeated until $f \leqslant 0$ is shown. As $f$ (or its derivatives) may not always be monotone on the whole domain, one has to distinguish several different cases and sub-cases. E.g., in the step-by-step version of the proof [21], following cases of Lemma 7 (i) are shown separately:

1. $c \geqslant a \geqslant b \geqslant sc$,
2. $c \geqslant a \geqslant b$, $b \leqslant sc$,
3. $c \geqslant a$, $a \leqslant b$,
4. $a \geqslant c$, $b \leqslant 2sc$, $sc \geqslant a - b$,
5. $a \geqslant c$, $b \geqslant 2sc$,
6. $a \geqslant c$, $b \leqslant 2sc$, $sc \leqslant a - b$.

Each case may have additional sub-cases that are distinguished. It is not obvious at first which cases should be distinguished and which partial derivatives should be taken. The choices in [21] were found by guidance via numerical evaluation of $f$ and its partial derivatives.

We do not go into further detail here, as the execution of the method described above does not seem to be of additional mathematical interest, but refer to [21] for the step-by-step proof.

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