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A synergetic perspective of analysing phase space dynamics: the crucial role of recurrence

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Abstract Hermann Haken's synergetics provides a fundamental theoretical framework for understanding the emergence of macroscopic order from microscopic interactions in complex dynamical systems. In this paper, we explore how different ways of quantifying recurrences serve as powerful tools to analyse phase-space dynamics within this synergetic perspective. Recurrence-based methods uncover evolving patterns, transitions between regular and chaotic behaviour, and characteristic time-scale separations that govern complex dynamics. We further discuss recent advances in combining recurrence analysis with machine learning, highlighting their potential to uncover hidden dynamical patterns and inform predictive modelling. Overall, recurrence approaches can be regarded as a synergetically motivated avenue for studying the universal properties of non-linear systems across disciplines.

1 Introduction

The study of complex dynamical systems has long been a central topic in non-linear science, with applications spanning physics, Earth sciences, biology, neuroscience, and economics [5, 8, 76, 80, 85, 105]. Hermann Haken's synergetics provides a powerful theoretical framework to understand self-organisation and emergent order in such systems [38]. According to synergetics, macroscopic structures arise from the interplay of microscopic degrees of freedom, with a few order parameters governing large-scale behaviour. The emergence of these structures often results from instabilities that drive system transitions, making the investigation of phase-space dynamics crucial for understanding the underlying mechanisms [34, 92, 96, 98, 109].

A key aspect of synergetics is the reduction of even high-dimensional systems to rather low ones, where slow, dominant modes dictate the evolution of fast, microscopic fluctuations. This principle makes phase-space analysis particularly insightful. To characterise these transitions, quantify different kinds of chaos, and identify underlying structures, different concepts have been introduced, like Lyapunov exponents, correlation dimension, and recurrence plots [28, 36, 110].

Lyapunov exponents measure the sensitivity of trajectories to initial conditions, providing insights into system stability and chaotic behaviour. However, they enable only to identify transitions from regular to chaotic regimes, but not chaos–chaos transitions which are also generated in manifold systems [2, 37, 51, 84]. Correlation dimension quantifies the effective degrees of freedom in the system, linking directly to Haken's concept of order parameters.

Recurrence plots (RPs) offer a very insightful visual approach to a system's dynamics. The corresponding statistical means enable us to detect complex structure, ordered dynamics, and even chaos-chaos transitions in phase space. This way we can investigate how self-organisation emerges from microscopic interactions and how instabilities lead to new macroscopic patterns.

This paper demonstrates how Haken's synergetic approach motivates modern non-linear analysis techniques based on phase space used for studying the dynamics of even high-dimensional complex systems. We especially discuss the crucial role of recurrence and summarise recent advances in combining it with machine learning (ML) approaches.

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2 Deterministic chaos

A main discovery in non-linear dynamics was that many non-linear systems can be highly sensitive to initial conditions, implying that long-range forecasting can fail even for purely deterministic systems. This has led to a new paradigm in science that was first scientifically shown by Henri Poincaré for conservative systems described as:

> "If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at the succeeding moment. But even if it were the case that the natural laws no longer had any secret for us, we could still only know the initial situation *approximately*. If that enabled us to predict the succeeding situation *with the same approximation*, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; small differences in the initial conditions can produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon". [83]

In contrast to the widely accepted paradigm of strong causality—embodied by Laplace's demon—this proposes a new paradigm of weak causality.

This erratic-like behaviour of many deterministic complex systems is strongly connected to the time span, after which the system's phenomenon trajectory comes back to a certain former point of its past, called recurrence time. The main property of this was also found by Poincaré 1885 in his recurrence theorem:

> "Suppose that a point P in phase space is covered by a conservative system. Then there will be trajectories which traverse a small surrounding of P infinitely often. That is to say, in some future time the system will return arbitrarily close to its initial situation and will do so infinitely often". [82]

This new type of dynamics was first called *chaos* by Li & Yorke in 1975 [62] and later more instructively *deterministic chaos* by Schuster 1984 [90]. This term very originally describes the complex interaction or regularity and randomness generated by such systems. Moreover, depending on their parameters, such complex systems exhibit regular regimes (periodic and quasi-periodic behaviour) as well as a variety of transitions between different regimes, for example, the classic routes to chaos. This rich dynamics is a basic source of self-organisation, in particular synergetics.

The action of non-linear dynamical systems is well expressed in an n-dimensional phase space, where they create a rich variety of forms, often of incredible beauty. n is the dimension of the system

$$\dot{\vec{x}}(t) = f(\vec{x}(t)) \tag{1}$$

where $\vec{x} \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$

There are basic parameters to distinguish these different regimes, especially the spectrum of Lyapunov exponents (LEs), $\lambda_1, \ldots, \lambda_n$, and Kolmogorov entropies. LEs measure how initially small perturbations grow with time:

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log \frac{\|\delta\vec{x}_i(t)\|}{\|\delta\vec{x}_i(0)\|} \tag{2}$$

where $\delta\vec{x}_i(t)$ is the i-th perturbation vector at time t and $\delta\vec{x}_i(0)$ is its initial magnitude.

If $\lambda_{\max} > 0$, the system is in a chaotic regime. The whole spectrum of the LEs provides even more information on the structure of the system (cf. [26]), enabling also a distinction of periodic, quasi-periodic, and fixed point regimes.

However, it does not indicate chaos–chaos transitions or intermittency, which are also basic features of such systems [37, 84]. Recurrence-based approaches offer techniques to identify these transitions as well, as will be explained in Sect. 4.

It should be emphasised that such complex systems must be non-linear (as most systems are in the real world) and, therefore, solutions of them as well as of their characteristic parameters cannot be found in analytical expressions in most cases. Hence, simulations are necessary, also when the equations are given, i.e. we have a similar situation as in case of experimental or observational data.

In contrast to simulated data from n-dimensional equations, we have typically only a very few and often only one parameter measured in experiments or observations. To transform such a time series $u(t)$ to an m-dimensional (pseudo-) phase space, an embedding procedure can be used [60, 81, 97]

$$\vec{x}(t) = (u(t), u(t-\tau), \ldots, u(t-(m-1)\tau)). \tag{3}$$

We can estimate appropriate m and τ via false nearest neighbours, and auto-correlation or mutual information [50, 55].

Some efficient techniques for a reliable estimation of the Lyapunov spectrum have been proposed [49, 88, 110]. However, in many practical applications as in neuroscience, climate, or fluid dynamics, we meet the "curse of dimensionality", i.e. the phase-space dimensions are too high for a reasonable analysis by these methods [27, 89]. Modern approaches try to overcome these limits, e.g. by combining phase-space reconstruction with ML models, such as the model adaptive phase-space reconstruction (MAPSR) method, which unifies the process of phase space reconstruction with the modelling of the dynamical system [22].

Whilst classical invariants such as Lyapunov exponents provide a rigorous characterisation of dynamical regimes as presented in their phase space, their reliable estimation becomes increasingly difficult for high-dimensional or short observational data. In such situations, it is instructive to return to the fundamental property already recognised in the early development of dynamical systems theory—*recurrence*. This property does neither rely on the explicit knowledge of governing equations nor on low-dimensional representations, and, thus, remains accessible even in complex empirical systems. Several approaches, therefore, build directly on recurrence behaviour rather than on asymptotic invariants, such as return time statistics or space-time separation plot [3, 42]. One particularly powerful framework in this context is provided by recurrence plots, which offer a way to visualise and quantify recurrences in the phase space reconstructed from observational data (Sect. 4).

3 Self-similarity and scaling

In addition to the approaches referring mainly to the dynamical behaviour of such systems, there is also a more geometric-based description. It analyses which kind of geometric patterns are generated from the corresponding trajectories in phase space. This has led to another paradigm in complex systems science: *self-similarity*. This means—generally speaking—that each magnification of typical details of the object looks like the whole (Julia sets, etc.). This property can be characterised by power-law-like scaling properties which are directly related to *fractal dimensions* of strange attractors [35, 40].

One main approach for their estimates from model systems, as well as experimental data is via the correlation sum (Fig. 1, [36]) of embedded time series, Eq. (3),

$$C_2 = \frac{1}{N(N-1)} \sum_{i \neq j=1}^{N} \Theta(\varepsilon - \|\vec{x}_i - \vec{x}_j\|), \tag{4}$$

where $\vec{x}_i$ are phase-space vectors (the states of the system, Eq. (1) or Eq. (3), with $i = 1, \ldots, N$ the indices of the observations), ε a threshold, and $\Theta(\cdot)$ the Heaviside function.

The correlation sum follows a power-law scaling $C_2(\varepsilon) \sim \varepsilon^{D_2}$ where the scaling exponent D_2 corresponds to the correlation dimension, a measure of the fractal geometry or effective dimensionality of the attractor. For example, a periodic time series yields a low dimension near 1, indicating a limit cycle, whilst the chaotic Lorenz system shows a higher dimension (a bit larger than 2), reflecting more complex dynamics. Stochastic processes (autocorrelated or white noise) exhibit scaling exponents near 1, typical for noise, which do not scale further with embedding dimension.

The correlation dimension measures how densely a complex object fills the phase space and can be a fractional number, reflecting the concept of a "strange attractor". Complementarily, lacunarity quantifies the gaps within such complex objects, characterising their heterogeneity [64]. This measure enables the description of multi-scale or multifractal systems, capturing even more intricate forms of self-organisation observed in various applications. We will provide further details when discussing recurrence plot analysis.

It is important to emphasise that both of these characteristics can be directly estimated via recurrence plots (Sect. 4).

4 Recurrence plots

Recurrence plots (RPs) combine both aspects, studying deterministic chaos and revealing self-similarity. The idea of the RP was introduced 1987 by Eckmann et al. [28] as a basic visual tool to investigate the phase-space dynamics of a system [72], derived from the recurrence matrix $\mathbf{R}(\varepsilon)$ with its entries:

$$R_{i,j}(\varepsilon) = \Theta(\varepsilon - \|\vec{x}_i - \vec{x}_j\|), \tag{5}$$

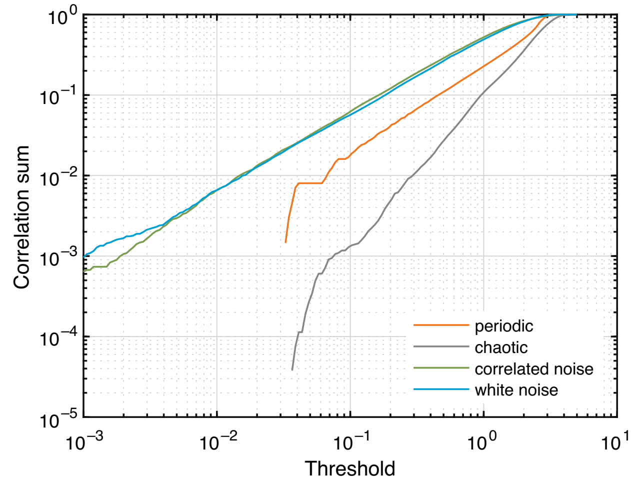

Fig. 1 Scaling of the correlation sum with recurrence threshold ε for time series presenting different types of dynamics: periodic, chaotic (Lorenz, x-component), autocorrelated noise, and white uniform noise (time series are shown in Fig. 2A). The corresponding scaling exponents (the correlation dimensions) are $D_2 = 1.05$ for periodic (deviation from 1.0 is due to the shortness of the considered time series), $D_2 = 2.02$ for Lorenz, and approximately 1.0 for the stochastic processes (where this value would scale with the embedding dimension). The time-series length used for the calculations was $N = 250$. The periodic time series was embedded using $m = 2$ and $\tau = 7$ (corresponding to a quarter of the period), the Lorenz x-component using $m = 3$ and $\tau = 10$, and the stochastic signals were not embedded

(using the same symbols and notation as defined in Eq. (4). The visual impression of the RP allows first conclusion about the dynamical system, even for high-dimensional systems [28], such as type of dynamics or regime transitions (Fig. 2B). Periodic dynamics yield regular diagonal lines in an RP, chaotic dynamics give fragmented diagonals with occasional regular blocks, white-noise signals cause a homogeneous pattern of isolated points, and temporally correlated (coloured) noise produces larger, less homogeneous clusters of recurrence points.

It quickly became apparent that the visible small-scale structures in an RP could be used to further characterise the systems quantitatively [66, 107, 114]. The density of recurrence points in the matrix $\mathbf{R}(\varepsilon)$, the diagonal and vertical line structures [70], as well as specific motifs [17, 44] in $\mathbf{R}(\varepsilon)$ represent characteristic features of the dynamical system.

The density of recurrence points, also known as the recurrence rate,

$$RR = \frac{1}{N^2} \sum_{i,\,j} R_{i,\,j}(\varepsilon) \tag{6}$$

coincides with the correlation sum [36] (Sect. 3). Its scaling with ε can be used to estimate the dimensionality of the dynamics, an important geometric property of the attractor in phase space (Fig. 1). This scaling can also be interpreted, within limits, as representing the self-similar structure of the attractor, one of the key principles in Haken's synergetics concept.

That the RP contains all important information on the dynamics can also be seen through the connection between the lengths of diagonal lines in the RP and the K_2 entropy and the correlation dimension D_2 [71]:

$$p_c(\varepsilon,\, l) \sim \varepsilon^{D_2} \mathrm{e}^{-l \Delta t\, K_2}, \tag{7}$$

where $p_c(\varepsilon,\, l)$ is the probability of finding a diagonal line of at least length l in the RP (this probability differs from the probability of finding a line of exactly length l, which is used in classical RQA).

An alternative method for capturing self-similarity in the recurrence of dynamical systems has been introduced recently, by combining the concept of lacunarity introduced by Mandelbrot [64] with recurrence analysis [9, 13, 65]. Recurrence lacunarity investigates the heterogeneity of the recurrence matrix across multiple scales, complementing the scaling analysis of the correlation sum discussed before. Whilst the recurrence rate captures the overall density of recurrence points, it does not distinguish between homogeneously distributed and strongly clustered structures that may share the same density. Lacunarity $\Lambda(r)$ fills this gap by analysing how the variance of recurrence point counts $n(r)$ within sliding boxes of increasing size r scales with the box size—asking not just how many recurrence

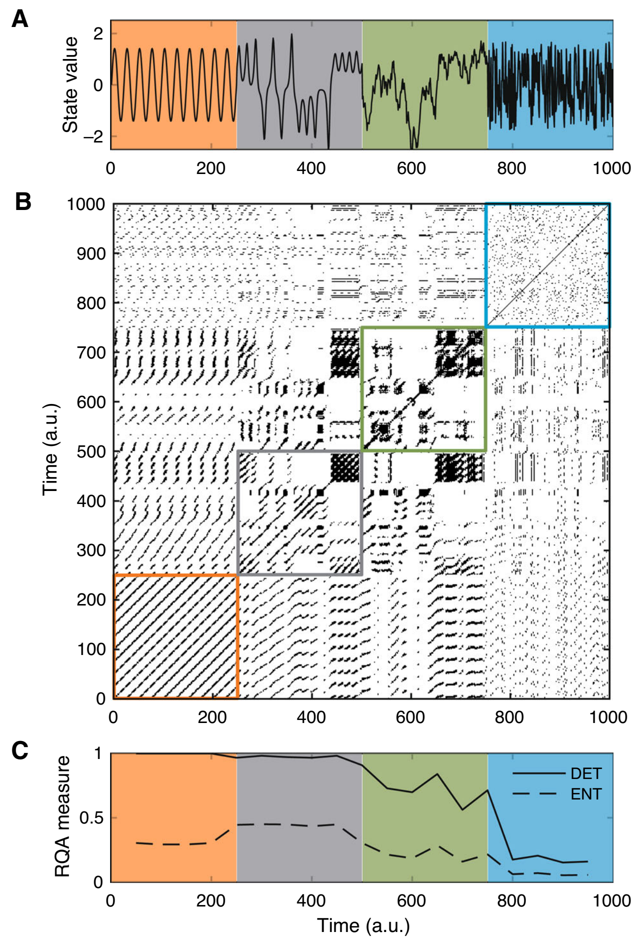

Fig. 2 **A** Time series presenting sequences of different types of dynamics: periodic, chaotic (Lorenz, x-component), autocorrelated noise, and white uniform noise (indicated by differently coloured background). The time series are concatenated only to visually compare different dynamical regimes in a compact way; however, similar situations of changing dynamics may arise in real-world time series within a single observation. **B** Recurrence plot of the time series, indicating the different types of dynamics, during the evolution of time (embedding with dimension $m = 3$ and delay $\tau = 4$, recurrence threshold ε selected to ensure a recurrence rate of 10%). It can also be considered for identifying regime transitions. In the chaotic dynamics, episodes of regular dynamics emerge, visible by the blocks of periodic lines (around time 250–290, 360–400, and 440–500), but also between 290 and 360 on larger time scales. **C** A moving-window RQA (with a window size of 100 and a window moving step of 10) helps to find regime changes quantitatively: the periodic and chaotic dynamics are deterministic systems, indicated by large determinism (DET) values; whereas the stochastic systems (autocorrelated and white uniform noise) have lower DET values. The entropy measure (ENT) reveals the high structural complexity of the chaotic system by large ENT values, whereas the other systems show low complexity (ENT is calculated using the *censi* correction schema [59])

points exist, but how unevenly they are distributed across scales:

$$\Lambda(r) = \frac{\langle n(r)^2 \rangle}{\langle n(r) \rangle^2} - 1. \tag{8}$$

A homogeneous recurrence matrix, as produced by white noise, yields low lacunarity across all scales, whereas clustered structures characteristic of chaotic or intermittent dynamics produce distinctly larger values. Characteristic attractor features—unstable periodic orbits, laminar phases, or slow-fast time scale separations—leave distinct signatures as peaks or scaling changes at the corresponding scales (Fig. 3), directly linking the self-similar geometry of the attractor to the temporal organisation of recurrences and revealing the strong connection between the geometric structure of the attractor and the temporal dynamics of the system [9, 65, 77].

In dynamical systems, large-scale dynamics emerge from the complex interactions of microscopic degrees of freedom, through mechanisms such as self-organisation, instability propagation, and scale-invariant behaviour.

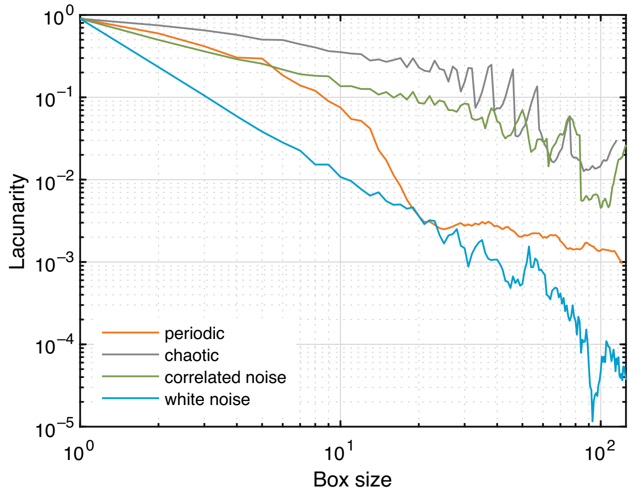

Fig. 3 Recurrence lacunarity for time series presenting different types of dynamics: periodic, chaotic (Lorenz, x-component), autocorrelated noise, and white uniform noise (time series are shown in Fig. 2A). **White noise** exhibits uniform RPs across all time scales, as shown by the lowest lacunarity values at each scale and the power-law scaling (the peak at very large box sizes results from the short length of the time series). The **periodic time series** has an RP with three different scaling behaviours: (1) as long as the box size is smaller than the line width (around 4 to 5, also visible by the small peak in the curve at this scale), the variability of patterns at this small scale is large, (2) box sizes exceeding the line width but smaller than the period of oscillation, reveal a steeper scaling, until (3) the box size exceeds the period of the oscillation (approximately 20) and now covers the entire variability of the periodic signal. The minimum at around 20 indicates the time scale of the oscillation. The **correlated noise** has more homogeneous patterns only at small scales and becomes more heterogeneous on larger scales, indicated by larger lacunarity values than for the periodic signal. The minimum around box size 100 indicates the overall large-scale and block-like typology of the RP of correlated noise as shown in Fig. 2B. The RP of the **chaotic time series** (Lorenz x-component) shows the largest variability on all scales (large lacunarity values). The peaks at box sizes larger than 20 indicate characteristic time scales of the patterns in the RP, such as return times, groups of line structures, and the blocks of periodic patterns (unstable periodic orbits). The time-series length used for the calculations was $N = 250$. The periodic time series was embedded using $m = 2$ and $\tau = 7$ (corresponding to a quarter of the period), the Lorenz x-component using $m = 3$ and $\tau = 10$, and the stochastic signals were not embedded. The recurrence threshold was selected to have a fixed recurrence rate of 10% for all time series

This can often be observed as characteristic patterns at different temporal scales. RPs can reveal such patterns, e.g. by blocks of periodic lines representing unstable periodic orbits [48, 71] or blocks of recurrences representing intermittency [58, 70]. The latter allows us to use RPs to indicate different types of chaos–chaos transitions [37, 70, 84]. The interplay of multiple time-scales manifesting in slow-fast dynamics is directly visible in RP by unique micro-patterns and clustered structures [53]. For a separation of scales, different multi-scale approaches for recurrence analysis have been developed [14, 87, 112, 113].

An important extension allows for the quantitative description of these features: the framework of recurrence quantification analysis (RQA) [70, 107, 114], including quantification via complex networks [23, 24, 73]. The diagonal and vertical lines in an RP are straightforward to interpret, making the RQA measures derived from these lines equally comprehensible. Temporal variations in these RQA measures identify changes in the dynamics of the systems [8, 70, 103] (Fig. 2C), although testing the significance of these variations is pivotal [67, 74]. For example, moving-window determinism (DET) and entropy (ENT) can quantify specific aspects of the RP structures over time (Fig. 2C). High DET values correspond to predictable deterministic regimes (periodic and chaotic), whereas the noise processes have lower DET, indicating less predictable dynamics. The entropy measure reveals the complexity of the chaotic system through elevated ENT values, whilst periodic and stochastic systems show lower complexity. From the perspective of synergetics, these variations indicate the emergence of order parameters that drive a system's macroscopic evolution. Just as dominant modes can control large-scale structures, RQA can detect shifts in these dynamical patterns and reveal transitions between order and chaos and even between chaos and chaos and intermittency. Thus, structures in RPs serve as an empirical signature of self-organisation.

RPs and RQA have demonstrated their power in unveiling complex dynamical patterns and transitions across a wide range of systems. Building on these foundations, recent advances enable further improvements in phase-space analysis through the combination of recurrence analysis with machine learning techniques. This opens new horizons for automated pattern recognition, enhanced prediction, and deeper insights into self-organisation phenomena. In the next section, we review emerging approaches that enhance the role of RPs in data-driven non-linear dynamics.

5 Integrating recurrence plots with machine learning

As a next step in the methodological evolution of recurrence analysis, the integration of machine learning (ML) with RPs has recently gained significant attention [45, 69]. Traditional RQA requires researchers to decide which measure is most informative for the studied system, or find effective ways to detect their common behaviour. Some systems might not even display distinctive line structures, favouring approaches like recurrence microstates [17] or the recurrence lacunarity approach described above. ML techniques can complement the existing RP approaches in tackling both of these challenges: they excel at compressing large feature vectors into lower-dimensional vectors, revealing the shared information across various RQA measures. They are also capable of learning higher-order patterns beyond line structures, leveraging RQA to uncover hidden patterns, predict emergent behaviour, and refine the order parameters governing system evolution. As a consequence, there is a rapidly growing body of scientific literature in which RPs and ML are combined.

Two complementary routes dominate this literature. The first treats the RP as a dynamical "snapshot" and feeds it to convolutional neural networks (CNNs) or, more recently, vision transformers [79]; the second extracts RQA-based feature vectors and passes them to learners suited to tabular data. CNNs are naturally suited to RPs because their successive convolutional and pooling layers of increasing receptive field size extract information across spatial scales, in a manner reminiscent of the boxes used for the computation of recurrence lacunarity (Fig. 3). Early layers detect local recurrence textures such as isolated points, short diagonal or vertical lines, and small-scale recurrence microstates; deeper layers capture recurrence clusters, periodic blocks, and the checkerboard textures characteristic of quasi-periodic dynamics; the deepest layers integrate recurrence macrostructures that may originate from regime shifts or intermittent chaos. This hierarchical decomposition mirrors the vocabulary of RQA, allowing the network to discover which structural motifs are diagnostically meaningful. The image route has been used for activity recognition [32], computer-aided diagnosis of medical signals [75, 115], anomaly detection in multisensor signals [15], and the distinction of dynamical classes, such as chaos versus noise [78]. Direct comparisons showed that the image-based method outperforms the alternative when applied to EEG data [57]. Generalising beyond individual systems, residual networks pre-trained on RPs of classical dynamical systems have been shown to transfer to novel systems, suggesting that RP-CNNs accumulate transferable dynamical knowledge [116]. Building on the standard RP, a non-linear spectrogram generated from the τ-recurrence rate has been shown to outperform Fourier spectrograms [41], and RPs can be combined with other time-series imaging techniques such as Gramian angular matrices (GAMs) through image fusion [46]. In this sense, the success of these approaches shows that RPs encode a hierarchy of dynamical pattern motifs that the network can learn to read as a multi-scale grammar of system dynamics.

Together, RPs and ML approaches can also be leveraged to study the self-organisation of dynamical systems. In fact, recent work shows that an ML model itself can be regarded as a self-organising dynamical system, with RP and RQA serving as diagnostic instruments of its internal state. In this context, reservoir computing has emerged as the most natural approach: echo state networks have been characterised from an explicit complex systems perspective, with their internal dynamics inspected through recurrence-based descriptors [6]. When trained on chaotic systems, reservoirs reproduce the target's recurrence structure on both temporal and spatial scales [108]. The reservoir's own state space has been promoted to a universal indicator of change in this framework, benchmarked explicitly against RP and RQA features and applied to concept-drift detection in non-stationary dynamical systems [100, 101]. Moreover, RPs help to elucidate how noise affects (hardware) reservoirs [29]. In each case, the trained network and the observed system function as two coupled, self-organised dynamical objects that exhibit mutual recurrences.

For the slow-fast structure often occurring in multi-scale dynamics, several works have explored a "decomposition-then-recurrence" approach: a signal is first split across scales using wavelets or similar techniques, an RP is computed at each scale, and a learner aggregates the resulting features [25, 52]. Even closer to the synergetic perspective are approaches that use RPs together with deep-learning architectures to recover the slow component of a forced system from observations dominated by fast dynamics. In particular, a CNN trained on the distance matrix (sometimes called "unthresholded RP") has been used to infer slow driving forces in systems with symmetry and almost-invariant subsets [43], and RQA-partitioned recurrence networks combined with kernel principal component analysis have extracted dynamical embeddings of low-frequency variability from atmospheric circulation patterns [77]. These studies illustrate how recurrence patterns can serve as a substrate on which ML separates slow from fast variables, reflecting the spirit of the synergetic reduction to slow modes.

Order parameters and phase transitions can usually not be extracted directly from real-world data. Consequently, studies instead track transitions in measurable time series properties. Recurrence quantifiers have long been deployed as early warning indicators of critical transitions. Their integration with ML classifiers is now an established line, especially in the study of thermoacoustic systems, where RQA features have been combined with support vector machines, k-means clustering, and random forests to detect the onset of instabilities [4, 30, 33, 54, 106]. Related work has used recurrence-based detection of regime drifts to flag the onset of disruptions in tokamak plasmas [18]. In parallel, an increasingly popular methodological strand uses recurrence microstates as compressed, order parameter-like features for ML [93, 94], and RQA features have been benchmarked systematically for anticipating dynamical state shifts between periodic, chaotic, and hyperchaotic regimes [99]. A topological data analysis component has been introduced by combining RPs with persistent homology to extract topological invariants suitable for downstream learning [47, 111]. Besides transitions between stable states, some systems exhibit metastable dynamics, a phenomenon which also Haken has significantly contributed to [39]. Recent work has shown that RP analysis could identify metastable states in complex time series and derive features that are useful for machine-learning-based classification tasks [20]. For multivariate systems, cross-recurrence analysis and ML approaches can also be applied together to characterise synchronisation patterns [91, 102], and recent work framed explicitly in order-parameter terms continues this research avenue [63]. Returning to the neural network itself as the self-organising system under study, RQA and ML have likewise been used to identify synchronisation transitions, where the synchronised state acts as an order parameter governing the remaining degrees of freedom [7, 11, 61].

Finally, both routes can be leveraged for prediction and equation discovery. An early precursor used recurrence-based partitioning of the phase space into near-stationary segments to fit local Gaussian-process predictors for non-linear and non-stationary systems [12], and applied extensions range from deep-learning load forecasting from RPs [86] to chaotic-system prediction via CNN-LSTM and multimodal architectures benchmarked on Lorenz and Rössler systems [56]. A conceptually rich development is the recent demonstration that the latent space of an encoder-decoder predictor is topologically equivalent to the underlying attractor, linking neural forecasting to Takens phase-space reconstruction [16]. Whilst direct equation discovery remains challenging, reservoirs trained on chaotic systems have been argued to constitute a viable alternative to explicit dynamical equations on the grounds that the recurrence structure of the trained reservoir is indistinguishable from that of the target [108], and recurrence-based descriptors have been used to characterise the empirical data-generating process of a time series [104].

Altogether, the combination of RPs and ML contributes to several key frontiers of synergetic research: multi-scale separation, order parameters, phase transitions, dynamical classification, and prediction. In these efforts, recurrence acts as a universal empirical signature of the low-dimensional, macroscopic structure that the synergetic programme set out to uncover. Explainability—a key concern in any field that is currently integrating ML—is starting to be actively addressed [46] but has the potential to be advanced further. Many of the aforementioned approaches come at high computational cost and may benefit from continuing advances in efficient RP computation [1, 19, 21, 31, 68, 95]. So far, largely absent from this otherwise active research field is any substantial integration of recurrence with data-driven equation discovery, which has come to dominate adjacent fields [10]. Three further directions appear similarly underdeveloped: causal inference from recurrence structure beyond bivariate measures, treatment of recurrence networks as graph neural networks, and large-scale pre-training of foundation models on RPs and should be studied in the future.

6 Conclusion

Hermann Haken's synergetics concept offers a powerful conceptual framework to understand the emergence of macroscopic order from the complex interactions of microscopic degrees of freedom in non-linear dynamical systems. This perspective emphasises the role of order parameters and instabilities in governing large-scale phase-space dynamics. Recurrence plots (RPs) and their quantitative descriptors provide a unique and insightful toolset that aligns naturally with this synergetic viewpoint, enabling the detection and characterisation of self-organisation, regime transitions, and emergent structures in high-dimensional systems.

Through recurrence analysis, we observe how slow dominant modes can be extracted from complex data, reflecting the synergetic reduction of dynamics to essential order parameters. The combination of RPs with modern machine learning techniques further enhances our ability to uncover hidden dynamical patterns, to classify regimes, and predict transitions—thus bridging empirical data-driven methods with the theoretical principles of synergetics.

Overall, recurrence-based methods serve as an empirical window into the fundamental mechanisms described by synergetics, capturing the interplay between microscopic fluctuations and emergent macroscopic order. A continued integration of recurrence analysis, non-linear dynamics, and data-driven approaches promises to deepen our understanding of self-organising systems across diverse disciplines, staying true to Haken's vision of revealing universal principles underlying complex phase-space dynamics.

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Author contribution statement

All authors contributed to the conception and design of the study. N.M. implemented the code for and prepared the illustrative figures. All authors drafted, reviewed, and edited the manuscript and approved the final version for submission.

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Data availability Code used for this analysis and to reproduce the results and figures of this study are available at Zenodo: https://doi.org/10.5281/zenodo.20435512.

Declarations

Use of generative AI Generative AI tools were used for language editing and text refinement. All scientific content was written and verified by the authors.

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