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Optimal mean-square consensus for heterogeneous multi-agent system with probabilistic time delay

Fenglan Sun[1,2,3] | **Hao Li**[2] | **Wei Zhu**[1] | **Jürgen Kurths**[3,4,5]

[1] Key Lab of Intelligent Analysis and Decision on Complex Systems, School of Science, Chongqing University of Posts and Telecommunications, Chongqing, P. R. China

[2] Key Lab of Intelligent Air-Ground Cooperative Control for Universities in Chongqing, College of Automation, Chongqing University of Posts and Telecommunications, Chongqing, P. R. China

[3] Department of Complexity Science, Potsdam Institute for Climate Impact Research, Potsdam, Germany (Email: kurths@pik-potsdam.de)

[4] Institute of Physics, Humboldt University of Berlin, Berlin, Germany

[5] Centre for Analysis of Complex Systems, Sechenov First Moscow State Medical University, Moscow, Russia

Correspondence
Fenglan Sun, No.2, Chongwen Road, Nan'an District, Chongqing 400065, China.
Email: sunfl@cqupt.edu.cn

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Abstract

This paper studies the mean-square consensus for heterogeneous multi-agent systems with probabilistic time delay. Each agent in the system has an objective function and only knows its own objective function. Control protocols for the system both over the fixed and the switched weighted-balanced topologies are designed. The consensus state of agents' position can make the sum of objective functions minimum. By adopting probability statistics, stochastic process, matrix theory and some stability method, sufficient conditions for the consensus protocol are given. Several simulations are presented to illustrate the potential correctness of the results.

1 | INTRODUCTION

Consensus of multi-agent systems (MAS) has attracted more and more attention because of its wide application [1–15]. Due to the complexity of environment, there are many occasions that require various kinds of agents cooperating with each other to complete tasks that cannot be done by a single agent. When different kinds of agents complete a task together, because of the influence of external environment, communication conditions and the different properties of agents themselves, the dynamics of agents may be different in a system. In other words, practically, agents in the same system usually have different dynamics. Some agents only need to consider their position. However, for some others, it needs to consider not only the position but also the velocity, even the acceleration, which leads to the heterogeneous multi-agent systems (HMAS) [15–24]. Hence it is important to study the consensus problem of HMAS, which mainly refers to systems consisting of both first-order and second-order agents. By using the impulsive control method and designing the distributed hybrid active controller, Wang et al. [15] investigated the output formation-containment problem of interacted linear HMAS. Guo et al. [17] studied the mean-square consensus of continuous HMAS with communication noises. Jiang et al. [21] investigated the couple-group consensus for a class of discrete-time HMAS with communication and input time delays. Sun et al. [24] studied the finite-time consensus of HMAS. In practice, the communication network of systems is not always being fixed. Instead, it usually switches among different topologies [11, 13, 16, 22], [25–27]. In addition, time delay and many other uncertain factors are inevitably and usually occurs randomly [7, 12, 14, 21], which may affect the convergence of a system, even destroy the stability of a relatively stable system, so it is important to analyze the consensus problem of system in statistic or probability sense [17–19, 22]. Li and Zhang [18] studied the mean-square average consensus of continuous MAS with white noises. Tan et al. [19] investigated the mean-square consensus of leader-following stochastic MAS by the distributed event-triggered control technique. Mo et al. [22] studied the velocity-constrained mean-square consensus of HMAS over Markovian switching topologies and time delay. In order to save the control cost or reduce the energy consumption, the optimization problem for MAS becomes more and more important [28–37]. Some of the optimal consensus results are on first-order systems [29–31], some on second-order systems [32, 33], and some others are on higher-order and heterogeneous systems [34–36]. Optimal consensus of MAS with cost functions or objective functions was studied in [28, 32] and [34]. Sun et al. [28] studied the distributed optimization algorithms for two types of HMAS under an undirected and connected communication graph. That is, systems composed of both continuous-time and discrete-time dynamic agents, and systems composed of both first-order and second-order dynamic agents are considered. The optimal consensus for linear MAS was studied in [35], for discrete-time HMAS in [36], and the quadratic case was studied in [37]. However, as far as we know, the optimal consensus on HMAS with probabilistic time delay has not been studied, although heterogeneous systems and probabilistic time delay are very common in practice. Therefore, this paper studies the optimal mean-square consensus for HMAS with probabilistic time delay, which is very significant both in theory and practice. However, due to the different dynamics of agents in HMAS and the uncertainty of probability delay, the normal methods that analyse the homogeneous MAS and deterministic time delay are no longer effective. Because of this problem, the properties of convex function and infinitesimal operator are fully used in this work. The main contributions of this paper are as follows: (i) Mean-square consensus for HMAS with probabilistic time delay is studied. (ii) Each agent in the system has an objective function that is only known by the agent itself. Moreover, the consensus state of agents' position can make the sum of objective functions minimum. (iii) Both the fixed and switched weighted-balanced topologies are considered.

In the following paper, some necessary symbols, concepts and lemmas are given in Section 2, the main results are presented in Section 3, several examples are provided in Section 4, and the conclusions are derived in Section 5.

2 | PRELIMINARIES

In this paper, R denotes the real number set, $R^{m\times n}$ the set of real matrix with m-row and n-column, I_n the n-dimensional identity matrix, $\mathbf{1} = (1, \dots, 1)^T$ a column vector with all elements being 1, $\mathbf{0}$ a zero matrix with a proper dimension, $\mathrm{E}[\chi]$ the mathematical expectation of χ, and $H > 0$ indicates the matrix H is positive definite. C_2 function refers to the second-order continuous differentiable function. And other symbols are in the usual sense.

Denote $G = (\mathcal{V}, \Sigma, \mathcal{A})$ a weighted graph, $\mathcal{V} = \{v_1, \dots, v_n\}$ the node set with every node representing an agent, $\Sigma \subseteq \mathcal{V} \times \mathcal{V}$ the edge set of the graph G, $\Upsilon = \{1, \dots, n\}$ the node index set, and $\Upsilon_1 = \{1, \dots, m\}$, $\Upsilon_2 = \{m+1, \dots, n\}$. In the digraph G, an edge $(v_i, v_j) \in \Sigma$ means that the j^{th} agent can receive the information from the i^{th} agent directly, and a directed path from v_j to v_i is a sequence of the edges $(v_j, v_{j_1}), (v_{j_1}, v_{j_2}), \dots, (v_{j_k}, v_i)$. If there is a directed path between any two distinct nodes, the graph is connected. The neighbour set of agent i is $\mathcal{N}_i = \{j | (v_j, v_i) \in \Sigma\}$, $i \in \Upsilon$, $\mathcal{A} = [a_{ij}] \in R^{n\times n}$ the adjacency matrix, $a_{ij} > 0$ if $(v_i, v_j) \in \Sigma$, otherwise $a_{ij} = 0$, and $a_{ii} = 0$ for all $i \in \Upsilon$. $D = diag\{\sum_{j=1}^{n} a_{1j}, \dots, \sum_{j=1}^{n} a_{nj}\}$ is the degree matrix of the graph G and $L = D - \mathcal{A}$ the Laplacian matrix. A graph G is called weighted balanced if there are some $w_i > 0, w_j > 0$, such that the coupling weights of G satisfy $w_i a_{ij} = w_j a_{ji}$ for all $i, j \in \Upsilon$.

Consider the following HMAS with m $(m < n)$ first-order agents and $(n - m)$ second-order agents. Their dynamics are

$$\dot{p}_i(t) = u_i(t), i \in \Upsilon_1, \tag{2.1}$$

and

$$\begin{aligned} \dot{p}_i(t) &= q_i(t), \\ \dot{q}_i(t) &= u_i(t), \end{aligned} \quad i \in \Upsilon_2. \tag{2.2}$$

with $p_i(t) \in R$, $q_i(t) \in R$, and $u_i(t) \in R$ being the position, velocity, and the control input of agent i, respectively.

Practically, systems are often subjected to time delay and the delay is usually random. That is there are two events as following

Event (I) : There is time delay in the system.
Event (II) : There is no time delay in the system.

Define the following random variable

$$\theta(t) = \begin{cases} 1, & \text{Event (I) occurs,} \\ 0, & \text{Event (II) occurs.} \end{cases}$$

Let the mathematical expectation of $\theta(t)$ be $\mathrm{E}[\theta(t) = 1] = \tilde{\theta}$ with $0 \le \tilde{\theta} \le 1$. Then $\mathrm{E}[\theta(t) = 0] = 1 - \tilde{\theta}$. The objective of this work is to solve the mean-square consensus of systems (2.1) and (2.2) with the probabilistic time delay such that the consensus state makes the sum of objective functions $g(p) =$

$\sum_{i=1}^{n} g_i(p_i)$ minimum, where function $g_i(p_i)$ is the objective function of agent i and is only known by agent i, $i \in \Upsilon$.

Definition 2.1. Systems (2.1)-(2.2) are said to achieve the optimal mean-square consensus, if there is

$$\lim_{t\to\infty} \mathrm{E}[||p_i(t) - p^*||^2] = 0, \quad i \in \Upsilon,$$

$$\lim_{t\to\infty} \mathrm{E}[||q_i(t) - q_j(t)||^2] = 0, i \in \Upsilon_2,$$

and $g(p^*) = \min \sum_{i=1}^{n} g_i(p_i)$.

Next, several necessary assumptions and lemmas are given for the further analysis.

Assumption 2.1. *Assume that the function $g_i : R^n \to R, i \in \Upsilon$, is strictly convex and twice differentiable. That is, for any $x, y \in R$, there is*

$$g_i(\vartheta x + (1-\vartheta)y) \le \vartheta g_i(x) + (1-\vartheta)g_i(y), \quad 0 \le \vartheta \le 1.$$

If and only if $x = y$, the equation holds. And the second derivative of function g_i, $i \in \Upsilon$, exists.

Assumption 2.2. *For any $x, y \in R^n$, function $g_i : R^n \to R, i \in \Upsilon$, is a C^2 function and its partial differential on x_i, that is, $\frac{\partial g_i(x)}{\partial x_i}$ is globally Lipschitz.*

Denote $f_i(x) = \frac{\partial g_i(x)}{\partial x_i}$, $i \in \Upsilon$. Then there is a positive $\rho_i > 0$, such that

$$||f_i(x) - f_i(y)|| \le \rho_i ||x - y||, \quad \forall\ i \in \Upsilon.$$

Assumption 2.3. *The time delay $d(t)$ satisfies $0 \le d(t) \le d_0$ with the constant $d_0 > 0$.*

Lemma 2.1 ([38]). *If the vector function $\phi(t) \in R^N$ is differentiable and the matrix $\Theta \in R^{N\times N}$ is positive definite, then there is*

$$\begin{aligned} & d_0^{-1}[\phi(t) - \phi(t-d(t))]^T \Theta [\phi(t) - \phi(t-d(t))] \\ & \le \int_{t-d(t)}^{t} \dot{\phi}^T(s)\Theta\dot{\phi}(s)ds, \qquad t \ge 0, \end{aligned}$$

where $d(t)$ satisfies Assumption 2.3.

3 | MAIN RESULTS

3.1 | Systems over fixed topology

For systems (2.1) and (2.2), design the following control protocol for the first-order agents

$$\begin{aligned} u_i(t) = {} & \theta(t)\alpha \sum_{j\in\mathcal{N}_i} w_i a_{ij}(p_j(t-d(t)) - p_i(t-d(t))) \\ & + (1-\theta(t))\alpha \sum_{j\in\mathcal{N}_i} w_i a_{ij}(p_j(t) - p_i(t)) \\ & - \gamma f_i(p_i), \quad i \in \Upsilon_1, \end{aligned} \tag{3.1}$$

and the following protocol for the second-order agents

$$\begin{aligned} u_i(t) = {} & \theta(t)\alpha \sum_{j\in\mathcal{N}_i} w_i a_{ij}(p_j(t-d(t)) - p_i(t-d(t))) \\ & + (1-\theta(t))\alpha \sum_{j\in\mathcal{N}_i} w_i a_{ij}(p_j(t) - p_i(t)) \\ & - \beta q_i(t) - \gamma f_i(p_i), i \in \Upsilon_2, \end{aligned} \tag{3.2}$$

with $\alpha, \beta, \gamma > 0$ being the control gains.

Under the protocols (3.1) and (3.2), systems (2.1) and (2.2) can be written as

$$\begin{aligned} \dot{p}_i(t) = {} & \alpha\theta(t) \sum_{j\in\mathcal{N}_i} w_i a_{ij}(p_j(t-d(t)) - p_i(t-d(t))) \\ & + \alpha(1-\theta(t)) \sum_{j\in\mathcal{N}_i} w_i a_{ij}(p_j(t) - p_i(t)) \\ & - \gamma f_i(p_i), \quad i \in \Upsilon_1, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \dot{p}_i(t) = {} & q_i(t), \\ \dot{q}_i(t) = {} & \alpha\theta(t) \sum_{j\in\mathcal{N}_i} w_i a_{ij}(p_j(t-d(t)) - p_i(t-d(t))) \\ & + \alpha(1-\theta(t)) \sum_{j\in\mathcal{N}_i} w_i a_{ij}(p_j(t) - p_i(t)) \\ & - \beta q_i(t) - \gamma f_i(p_i), i \in \Upsilon_2. \end{aligned} \tag{3.4}$$

For simplicity, denote the Laplacian matrix as

$$L = \begin{bmatrix} L_{ff} & L_{fs} \\ L_{sf} & L_{ss} \end{bmatrix},$$

with

$$L_{ff} = \begin{bmatrix} l_{11} & \cdots & l_{1m} \\ \vdots & \ddots & \vdots \\ l_{m1} & \cdots & l_{mm} \end{bmatrix},$$

$$L_{fs} = \begin{bmatrix} l_{1(m+1)} & \cdots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{m(m+1)} & \cdots & l_{mn} \end{bmatrix},$$

$$L_{sf}=\begin{bmatrix} l_{(m+1)1} & \cdots & l_{(m+1)m}\\ \vdots & \ddots & \vdots\\ l_{n1} & \cdots & l_{nm}\end{bmatrix},$$

$$L_{ss}=\begin{bmatrix} l_{(m+1)(m+1)} & \cdots & l_{(m+1)n}\\ \vdots & \ddots & \vdots\\ l_{n(m+1)} & \cdots & l_{nn}\end{bmatrix}.$$

Let $p^*=(p_1^*,\dots,p_n^*)^T, q^*=(q_{m+1}^*,\dots,q_n^*)^T$ be the equilibrium point of systems (3.3) and (3.4). Then at this equilibrium point there is

$$\begin{aligned}&\alpha\theta(t)\sum_{j\in\mathcal{N}_i} w_ia_{ij}(p_j(t-d(t))-p_i(t-d(t)))\\&+(1-\theta(t))\alpha\sum_{j\in\mathcal{N}_i} w_ia_{ij}(p_j(t)-p_i(t))\\&-\gamma f_i(p_i)=0, i\in\Upsilon_1,\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}&q_i(t)=0,\\&\alpha\theta(t)\sum_{j\in\mathcal{N}_i} w_ia_{ij}(p_j(t-d(t))-p_i(t-d(t)))\\&+\alpha(1-\theta(t))\sum_{j\in\mathcal{N}_i} w_ia_{ij}(p_j(t)-p_i(t))\\&-\beta q_i(t)-\gamma f_i(p_i)=0, i\in\Upsilon_2.\end{aligned}\tag{3.6}$$

The sum of (3.5) and (3.6) from 1 to n is

$$\begin{aligned}&\alpha\theta(t)\sum_{i=1}^n\sum_{j=1}^n w_ia_{ij}(p_j(t-d(t))-p_i(t-d(t)))\\&+\alpha(1-\theta(t))\sum_{i=1}^n\sum_{j=1}^n w_ia_{ij}(p_j(t)-p_i(t))\\&-\beta\sum_{i=m+1}^n q_i(t)-\gamma\sum_{i=1}^n f_i(p_i)=0,\end{aligned}\tag{3.7}$$

and

$$\sum_{i=m+1}^n q_i(t)=0.\tag{3.8}$$

Note that $w_ia_{ij}=w_ja_{ji}, \forall\, i,j\in\Upsilon$. Then

$$\sum_{i=1}^n\sum_{j=1}^n w_ia_{ij}(p_j(t-d(t))-p_i(t-d(t)))\equiv 0,\tag{3.9}$$

and

$$\sum_{i=1}^n\sum_{j=1}^n w_ia_{ij}(p_j(t)-p_i(t))\equiv 0.\tag{3.10}$$

From (3.7) to (3.10) one can get, at the equilibrium point of systems (3.3) and (3.4), there is $\sum_{i=1}^n f_i(p_i)\equiv 0$. Thus this equilibrium point is optimal of the function $g(p)$, that is, $p_i^*=p^*,\ \forall\, i\in\Upsilon$, and $q_i^*=0, i\in\Upsilon_2$.

Let $\xi_i(t)=p_i(t)-p_i^*, i\in\Upsilon,\ \eta_i(t)=q_i(t)-q_i^*=q_i(t), i\in\Upsilon_2$. Then systems (3.3)-(3.4) can be written as

$$\begin{aligned}\dot\xi_i(t)=&\,\alpha\theta(t)\sum_{j\in\mathcal{N}_i} w_ia_{ij}(\xi_j(t-d(t))-\xi_i(t-d(t)))\\&+\alpha(1-\theta(t))\sum_{j\in\mathcal{N}_i} w_ia_{ij}(\xi_j(t)-\xi_i(t))\\&-\gamma\big(f_i(\xi_i(t)+p^*)-f_i(p^*)\big),\end{aligned}\tag{3.11}$$

and for $i\in\Upsilon_2$, there is

$$\begin{aligned}\dot\xi_i(t)=&\,\eta_i(t),\\ \dot\eta_i(t)=&\,\alpha\theta(t)\sum_{j\in\mathcal{N}_i} w_ia_{ij}(\xi_j(t-d(t))-\xi_i(t-d(t)))\\&+\alpha(1-\theta(t))\sum_{j\in\mathcal{N}_i} w_ia_{ij}(\xi_j(t)-\xi_i(t))\\&-\gamma\big(f_i(\xi_i(t)+p^*)-f_i(p^*)\big)-\beta\eta_i(t).\end{aligned}\tag{3.12}$$

Denote

$$\begin{aligned}y_1&=\big(\xi_1,\dots,\xi_m\big)^T,\\ y_2&=\big(\xi_{m+1},\dots,\xi_n\big)^T,\\ y_3&=\big(\eta_{m+1},\dots,\eta_n\big)^T,\\ \tilde g_1&=\big(f_1(\xi_1+p^*)-f_1(p^*),\\&\quad\dots,f_m(\xi_m+p^*)-f_m(p^*)\big)^T,\\ \tilde g_2&=\big(f_{m+1}(\xi_{m+1}+p^*)-f_{m+1}(p^*),\\&\quad\dots,f_n(\xi_n+p^*)-f_n(p^*)\big)^T,\end{aligned}$$

and $\tilde y(t)=(y_1^T,y_2^T,y_3^T)^T, \tilde g(t)=(\tilde g_1^T,0,\tilde g_2^T)^T\in R^{2n-m}$. Then systems (3.11) and (3.12) can be described as

$$\begin{aligned}\dot{\tilde y}(t)=&\,(\alpha(\theta(t)-1)M_1+M_2)\tilde y(t)\\&-\alpha\theta(t)M_1\tilde y(t-d(t))-\gamma\tilde g(t),\end{aligned}\tag{3.13}$$

where

$$M_1 = \begin{bmatrix} W_1 L_{ff} & W_1 L_{fs} & 0 \\ 0 & 0 & 0 \\ W_2 L_{sf} & W_2 L_{ss} & 0 \end{bmatrix} \in R^{2n-m},$$

$$M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_{n-m} \\ 0 & 0 & -\beta I_{n-m} \end{bmatrix} \in R^{2n-m},$$

$W_1 = diag\{w_1, \ldots, w_m\}$, $W_2 = diag\{w_{m+1}, \ldots, w_n\}$.

Remark 3.1 Note that if Assumption 2.2 holds and $\lim_{t\to\infty} \tilde{y}(t) = 0$, then $\lim_{t\to\infty} \tilde{g}(t) = 0$.

Theorem 3.1. *Suppose that the network is connected and weighted balanced, and Assumptions 2.1–2.3 hold. Then under the control protocols (3.1) and (3.2) with some* $\alpha, \beta, \gamma > 0$*, systems (2.1) and (2.2) can reach the optimal mean-square consensus with the objective function* $g(p) = \sum_{i=1}^{n} g_i(p)$*, if there exist symmetric matrices* $\mathcal{P} > 0$, $\mathcal{R} > 0$*, such that for certain* $d_0 > 0$*, the following linear matrix inequality*

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} \\ [6pt]\Psi_{31} & \Psi_{32} & \Psi_{33} \end{bmatrix} < 0, \tag{3.14}$$

holds where

$$\Psi_{11} = 2\alpha(\tilde{\theta} - 1)\mathcal{P}M_1 + 2\mathcal{P}M_2 + \alpha^2 d_0(\tilde{\theta} - 1)^2 M_1^T \mathcal{R} M_1$$
$$+ \alpha d_0(\tilde{\theta} - 1)M_1^T \mathcal{R} M_2 + \alpha d_0(\tilde{\theta} - 1)M_2^T \mathcal{R} M_1$$
$$- d_0^{-1}\mathcal{R} + d_0 M_2^T \mathcal{R} M_2,$$

$$\Psi_{12} = -\alpha\tilde{\theta}\mathcal{R}M_1 - \alpha^2 d_0\tilde{\theta}(\tilde{\theta} - 1)M_1^T \mathcal{R} M_1$$
$$- \alpha d_0\tilde{\theta} M_2^T \mathcal{R} M_1 + d_0^{-1}\mathcal{R},$$

$$\Psi_{13} = -\gamma\mathcal{P} - \gamma\alpha d_0(\tilde{\theta} - 1)M_1^T \mathcal{R} - \gamma d_0 M_2^T \mathcal{R},$$

$$\Psi_{22} = \alpha^2 d_0\tilde{\theta}^2 M_1^T \mathcal{R} M_1 - d_0^{-1}\mathcal{R},$$

$$\Psi_{23} = \gamma\alpha d_0\tilde{\theta}\mathcal{R}M_1, \quad \Psi_{33} = d_0\gamma^2\mathcal{R},$$

$$\Psi_{21} = \Psi_{12}^T, \quad \Psi_{31} = \Psi_{13}^T, \quad \Psi_{32} = \Psi_{23}^T.$$

Proof. Choose the candidate Lyapunov function

$$\mathcal{V}(\tilde{y}_t) = \mathcal{V}_1(\tilde{y}_t) + \mathcal{V}_2(\tilde{y}_t), \tag{3.15}$$

with

$$\mathcal{V}_1(\tilde{y}_t) = \tilde{y}^T(t)\mathcal{P}\tilde{y}(t),$$
$$\mathcal{V}_2(\tilde{y}_t) = \int_{t-d_0}^{t} (s - t + d_0)\dot{\tilde{y}}^T(s)\mathcal{R}\dot{\tilde{y}}(s)ds,$$

and $\tilde{y}_t$ is the short form of function $\tilde{y}(t)$, the same as other similar symbols in the following. The infinitesimal operator $\mathcal{L}$ of function $\mathcal{V}(\tilde{y}_t)$ is defined as

$$\mathcal{L}(\mathcal{V}(\tilde{y}_t)) = \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left\{ \mathrm{E}[\mathcal{V}(\tilde{y}_{t+\Delta t})|\mathcal{V}(\tilde{y}_t)] - \mathcal{V}(\tilde{y}_t)\right\}. \tag{3.16}$$

Then along (3.13) there is

$$\begin{aligned} \mathcal{L}(\mathcal{V}_1(\tilde{y}_t)) &= \tilde{y}^T(t)(2\alpha(\tilde{\theta} - 1)\mathcal{P}M_1 + 2\mathcal{P}M_2)\tilde{y}(t) \\ &+ \tilde{y}^T(t)(-2\alpha\tilde{\theta}\mathcal{P}M_1)\tilde{y}(t - d(t)) \\ &+ \tilde{y}^T(t)(-2\gamma\mathcal{P})\tilde{g}(t), \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} &\mathcal{L}(\mathcal{V}_2(\tilde{y}_t)) \\ &= d_0((\alpha(\tilde{\theta} - 1)M_1 + M_2)\tilde{y}(t) - \alpha\tilde{\theta}M_1\tilde{y}(t - d(t)) - \gamma\tilde{g}(t))^T \\ &\mathcal{R}((\alpha(\tilde{\theta} - 1)M_1 + M_2)\tilde{y}(t) \\ &- \alpha\tilde{\theta}M_1\tilde{y}(t - d(t)) - \gamma\tilde{g}(t)) \\ &- \int_{t-d_0}^{t} \dot{\tilde{y}}^T(s)\mathcal{R}\dot{\tilde{y}}(s)ds. \end{aligned} \tag{3.18}$$

According to Assumption 2.3 and Lemma 2.1 one can obtain

$$\begin{aligned} &\int_{t-d_0}^{t} \dot{\tilde{y}}^T(s)\mathcal{R}\dot{\tilde{y}}(s)ds \geq \int_{t-d(t)}^{t} \dot{\tilde{y}}^T(s)\mathcal{R}\dot{\tilde{y}}(s)ds \\ &\geq d_0^{-1}[\tilde{y}(t) - \tilde{y}(t - d(t))]^T\mathcal{R}[\tilde{y}(t) - \tilde{y}(t - d(t))]. \end{aligned} \tag{3.19}$$

Based on (3.18) and (3.19) there is

$$\begin{aligned} \mathcal{L}(\mathcal{V}_2(\tilde{y}_t)) &\leq d_0((\alpha(\tilde{\theta} - 1)M_1 + M_2)\tilde{y}(t) - \alpha\tilde{\theta}M_1\tilde{y}(t - d(t)) \\ &- \gamma\tilde{g}(t))^T\mathcal{R}((\alpha(\tilde{\theta} - 1)M_1 + M_2)\tilde{y}(t) \\ &- \alpha\tilde{\theta}M_1\tilde{y}(t - d(t)) - \gamma\tilde{g}(t)) \\ &- d_0^{-1}(\tilde{y}(t) - \tilde{y}(t - d(t)))^T\mathcal{R}(\tilde{y}(t) - \tilde{y}(t - d(t))) \\ &= \tilde{y}^T(t)(d_0(\alpha(\tilde{\theta} - 1)M_1^T + M_2^T)\mathcal{R}(\alpha(\tilde{\theta} - 1)M_1 \\ &+ M_2) - d_0^{-1}\mathcal{R})\tilde{y}(t) + \tilde{y}^T(t)(-\alpha d_0\tilde{\theta}(\alpha(\tilde{\theta} - 1)M_1^T \end{aligned}$$

$$+M_2^T)\mathcal{R}M_1 + d_0^{-1}\mathcal{R})\tilde{y}(t-d(t))$$
$$+\tilde{y}^T(t)(-\gamma d_0(\alpha(\tilde{\theta}-1)M_1^T + M_2^T)\mathcal{R})\tilde{g}(t)$$
$$+\tilde{y}^T(t-d(t))(\alpha d_0\tilde{\theta}M_1^T\mathcal{R}(\alpha(\tilde{\theta}-1)M_1 + M_2)$$
$$+d_0^{-1}\mathcal{R})\tilde{y}(t) + \tilde{y}^T(t-d(t))(d_0\alpha^2\tilde{\theta}^2M_1^T\mathcal{R}M_1$$
$$-d_0^{-1}\mathcal{R})\tilde{y}(t-d(t))$$
$$+\tilde{y}^T(t-d(t))(\gamma\alpha d_0\tilde{\theta}M_1^T\mathcal{R})\tilde{g}(t)$$
$$+\tilde{g}^T(t)(-\gamma d_0\mathcal{R}(\alpha(\tilde{\theta}-1)M_1 + M_2))\tilde{y}(t)$$
$$+\tilde{g}^T(t)(\gamma\alpha d_0\tilde{\theta}\mathcal{R}M_1)\tilde{y}(t-d(t))$$
$$+\tilde{g}^T(t)(\gamma^2 d_0\mathcal{R})\tilde{g}(t). \tag{3.20}$$

Then from (3.15), (3.17)–(3.20) one can get

$$\begin{aligned}
\mathcal{L}(\mathcal{V}(t)) &= \mathcal{L}(\mathcal{V}_1(t)) + \mathcal{L}(\mathcal{V}_2(t)) \\
&\le \tilde{y}^T(t)(2\alpha(\tilde{\theta}-1)\mathcal{R}M_1 + 2\mathcal{P}M_2 \\
&+ \alpha^2 d_0(\tilde{\theta}-1)^2M_1^T\mathcal{R}M_1 + \alpha d_0(\tilde{\theta}-1)M_1^T\mathcal{R}M_2 \\
&+ \alpha d_0(\tilde{\theta}-1)M_2^T\mathcal{R}M_1 + d_0M_2^T\mathcal{R}M_2 - d_0^{-1}\mathcal{R})\tilde{y}(t) \\
&+ \tilde{y}^T(t)(-\alpha\tilde{\theta}\mathcal{P}M_1 - \alpha^2 d_0\tilde{\theta}(\tilde{\theta}-1)M_1^T\mathcal{R}M_1 \\
&- \alpha d_0\tilde{\theta}M_2^T\mathcal{R}M_1 + d_0^{-1}\mathcal{R})\tilde{y}(t-d(t)) \\
&+ \tilde{y}^T(t)(-\gamma P - \gamma\alpha d_0(\tilde{\theta}-1)M_1^T\mathcal{R} \\
&- \gamma d_0M_2^T\mathcal{R})\tilde{g}(t) + \tilde{y}^T(t-d(t))(-\alpha\tilde{\theta}M_1^T\mathcal{P} \\
&- \alpha^2 d_0\tilde{\theta}(\tilde{\theta}-1)M_1^T\mathcal{R}M_1 \\
&- \alpha d_0\tilde{\theta}M_1^T\mathcal{R}M_2 + d_0^{-1}\mathcal{R})\tilde{y}(t) \\
&+ \tilde{y}^T(t-d(t))(d_0\alpha^2\tilde{\theta}^2M_1^T\mathcal{R}M_1 \\
&- d_0^{-1}\mathcal{R})\tilde{y}(t-d(t)) \\
&+ \tilde{y}^T(t-d(t))(\gamma\alpha d_0\tilde{\theta}M_1^T\mathcal{R})\tilde{g}(t) \\
&+ \tilde{g}^T(t)(-\gamma\mathcal{R} - \gamma\alpha d_0(\tilde{\theta}-1)\mathcal{R}M_1 \\
&- \gamma d_0\mathcal{R}M_2))\tilde{y}(t) \\
&+ \tilde{g}^T(t)(\gamma\alpha d_0\tilde{\theta}\mathcal{R}M_1)\tilde{y}(t-d(t)) \\
&+ \tilde{g}^T(t)(\gamma^2 d_0\mathcal{R})\tilde{g}(t).
\end{aligned} \tag{3.21}$$

Let $\tilde{Y} = [\tilde{y}^T(t)\ \tilde{y}^T(t-d(t))\ \tilde{g}^T(t)]^T$. Then (3.21) can be written as

$$\mathcal{L}(\mathcal{V}(\tilde{y}_t)) \le \tilde{Y}^T\Psi\tilde{Y}, \tag{3.22}$$

which implies

$$\mathrm{E}[\mathcal{L}(\mathcal{V}(\tilde{y}_t))] \le \mathrm{E}[\tilde{Y}^T\Psi\tilde{Y}] = \mathrm{E}[\tilde{Y}]^T\Psi\mathrm{E}[\tilde{Y}]. \tag{3.23}$$

According to (3.14) and (3.23), one can obtain $\mathrm{E}[\mathcal{L}(\mathcal{V}(\tilde{y}_t))] < 0$. Then the original of system (3.13) is mean-square stability, which implies that $\lim_{t\to\infty}\mathrm{E}[||\tilde{y}^2(t)||] = 0$. This together with systems (3.11) and (3.12) gives $\lim_{t\to\infty}\mathrm{E}[||\xi(t)||^2] = 0$ and $\lim_{t\to\infty}\mathrm{E}[||\eta(t)||^2] = 0$. That is, $\lim_{t\to\infty}\mathrm{E}[||p_i(t) - p^*||^2] = 0$ and $\lim_{t\to\infty}\mathrm{E}[||q_i(t) - q^*||^2] = 0$. Note that $g(p^*) = \min\sum_{i=1}^n g_i(p)$. The proof is completed. □

3.2 | Systems over switching topology

Due to the complexity of environment, in practice, the communication between agents usually varies. Hence this section considers systems (2.1) and (2.2) over the switching topologies $G_k, k \in \mathbb{N}$, which are connected and weighted balanced, $\mathbb{N} = \{1, 2, \dots, \aleph\}$ is the index set and $\aleph$ a finite positive integer. That is for the adjacency matrix $A_k = [a_{ij}^k]_{n\times n}$ of graph G_k, there exists w_i^k such that $w_i^k a_{ij}^k = w_j^k a_{ji}^k$, $k \in \mathbb{N}$.

For systems (2.1) and (2.2) over the switching weight balanced network $G_k, k \in \mathbb{N}$, which switches in sequence $G_1, G_2, \dots, G_\aleph$, design the following control protocol for the first-order agents

$$\begin{aligned}
u_i(t) &= \alpha\theta(t)\sum_{j\in\mathcal{N}_i} w_i^k a_{ij}^k(p_j(t-d(t)) - p_i(t-d(t))) \\
&+ \alpha(1-\theta(t))\sum_{j\in\mathcal{N}_i} w_i^k a_{ij}^k(p_j(t) - p_i(t)) \\
&- \gamma f_i(p_i), \qquad \forall\ k \in \mathbb{N},\ i \in \Upsilon_1,
\end{aligned} \tag{3.24}$$

and the following protocol for the second-order agents

$$\begin{aligned}
u_i(t) &= \alpha\theta(t)\sum_{j\in\mathcal{N}_i} w_i^k a_{ij}^k(p_j(t-d(t)) - p_i(t-d(t))) \\
&+ \alpha(1-\theta(t))\sum_{j\in\mathcal{N}_i} w_i^k a_{ij}^k(p_j(t) - p_i(t)) \\
&- \beta q_i(t) - \gamma f_i(p_i), \qquad \forall\ k \in \mathbb{N},\ i \in \Upsilon_2.
\end{aligned} \tag{3.25}$$

Then similar the analysis on the fixed topology, there is the following result.

Theorem 3.2. *Suppose that Assumptions 2.1–2.3 hold and the network is switched among the weighted balanced graph $G_k, k \in \mathbb{N}$. Then under the control protocols (3.24) and (3.25) with some $\alpha, \beta, \gamma > 0$, systems (2.1) and (2.2) can reach the optimal mean-square consensus with the objective function $g(p) = \sum_{i=1}^n g_i(p)$, if for any $k \in \mathbb{N}$, there exist symmetric matrices $\mathcal{P}_k > 0, \mathcal{R}_k > 0$, such that for certain $d_0 > 0$, the* *following inequalities*

$$\Psi_k = \begin{bmatrix} \Psi_{11}^k & \Psi_{12}^k & \Psi_{13}^k \\ \Psi_{21}^k & \Psi_{22}^k & \Psi_{23}^k \\ \Psi_{31}^k & \Psi_{32}^k & \Psi_{33}^k \end{bmatrix} < 0,$$

hold, where

$$\begin{aligned} \Psi_{11}^k &= 2\alpha(\tilde{\theta}-1)\mathcal{R}_k M_{1k} + 2\mathcal{P}_k M_{2k} \\ &\quad + \alpha^2 d_0(\tilde{\theta}-1)^2 M_{1k}^T \mathcal{R}_k M_{1k} - d_0^{-1}\mathcal{R}_k \\ &\quad + \alpha d_0(\tilde{\theta}-1)M_{1k}^T \mathcal{R}_k M_{2k} + d_0 M_{2k}^T \mathcal{R}_k M_{2k} \\ &\quad + \alpha d_0(\tilde{\theta}-1)M_{2k}^T \mathcal{R}_k M_{1k}, \\ \Psi_{12}^k &= -\alpha\tilde{\theta}\mathcal{P}_k M_{1k} - \alpha^2 d_0\tilde{\theta}(\tilde{\theta}-1)M_{1k}^T \mathcal{R}_k M_{1k} \\ &\quad - \alpha d_0\tilde{\theta}M_{2k}^T \mathcal{R}_k M_{1k} + d_0^{-1}\mathcal{R}_k, \\ \Psi_{13}^k &= -\gamma\mathcal{P}_k - \gamma\alpha d_0(\tilde{\theta}-1)M_{1k}^T\mathcal{R}_k - \gamma d_0 M_{2k}^T \mathcal{R}_k, \\ \Psi_{22}^k &= \alpha^2 d_0\tilde{\theta}^2 M_{1k}^T \mathcal{R}_k M_{1k} - d_0^{-1}\mathcal{R}_k, \\ \Psi_{23}^k &= \gamma\alpha d_0\tilde{\theta}\mathcal{R}_k M_{1k}, \quad \Psi_{33}^k = d_0\gamma^2\mathcal{R}_k, \\ \Psi_{21}^k &= (\Psi_{12}^k)^T, \ \Psi_{31}^k = (\Psi_{13}^k)^T, \ \Psi_{32}^k = (\Psi_{23}^k)^T. \end{aligned}$$

The proof of Theorem 3.2 is similar to the argument in Theorem 3.1. Hence it is omitted here.

Remark 3.2. According to the control protocols (3.1) and (3.2), and (3.24) and (3.25), $\tilde{\theta} \equiv 1$ implies the delay case, that is Event (I) occurs with probability one. And $\tilde{\theta} \equiv 0$ implies the delay free case, that is Event (II) occurs with probability one. Hence the delay system and the delay free system are the special cases of this work.

Remark 3.3. For simplicity, only one dimensional system is considered in this work. It should be pointed out that the result is also valid for the multi-dimensional system, which can be obtained by using the Kronecker product of matrix.

4 | EXAMPLES

Example. Consider a system with two first-order agents denoted as $i = 1, 2$, and three second-order agents denoted as $i = 3, 4, 5$. Choose the initial states $p(0) = (-6, -12, -3, 6, 12)^T$, $q(0) = (3, 10, -6)^T$. The adjacency

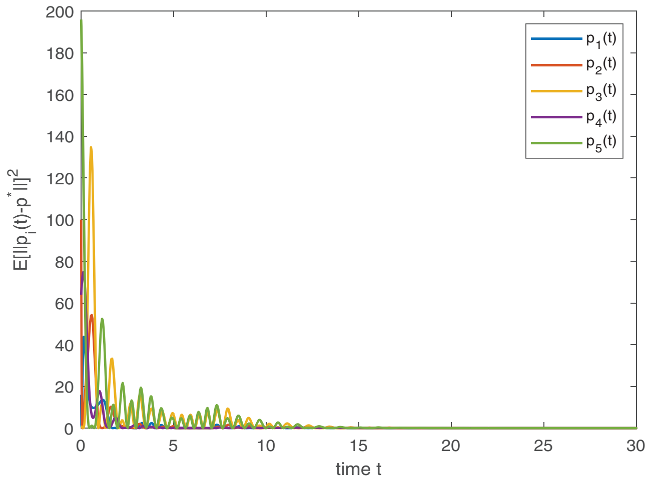

FIGURE 1 Mean value of agents' position error over G_1 with $d(t) = 0.02|\cos t|$ and $\tilde{\theta} = 0.3$

matrices of graphs G_1, G_2, G_3 are given as

$$A_1 = \begin{bmatrix} 0 & 2 & 0 & 0 & 2 \\ 3 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 3 \\ 2 & 0 & 0 & 2 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 3 & 3 & 0 & 3 & 0 \\ 2 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 3 & 0 & 0 & 3 \\ 2 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 2 & 2 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \end{bmatrix},$$

and the corresponding coupling coefficients are $\mathcal{W}_{A_1} = [w_{11}, w_{12}, w_{13}, w_{14}, w_{15}] = [3, 2, 3, 2, 3]$, *where* $w_{11}, w_{12}, w_{13}, w_{14}, w_{15}$ *are the coupling weight of graph* G_1. *Similarly,* $\mathcal{W}_{A_2} = \mathcal{W}_{A_3} = [2, 3, 2, 3, 2]$. *The objective functions* $g_i(p)$ *of agent* i *are given as* $g_1(p) = (p+1)^2 + 1$, $g_2(p) = (p+5)^2 + 3$, $g_3(p) = (p+3)^2 + 1$, $g_4(p) = (p-5)^2 - 1$, $g_5(p) = (p+6)^2$.

Note that graphs G_1, G_2 and G_3 are connected and weighted balanced, and $p^* = -2$ is the optimal point of the function $g(p) = \sum_{i=1}^5 g_i(p)$. That is $g(-2) = \min \sum_{i=1}^{5} g_i(p) = 80$. For simplicity, in all the simulations, the parameters are chosen as $\alpha = 3, \beta = 0.5, \gamma = 0.04$.

(i) For systems (2.1) and (2.2) over G_1, under protocols (3.1) and (3.2) with $\tilde{\theta} = 0.3$, $d(t) = 0.02|\cos t|$, by solving the linear matrix inequality (3.14) one can get $d_0 \leq 0.067$. Mean values of agents' position error and velocity error are given in Figures 1 and 2, which show that the mean values of all agents' states achieve together. Especially, the mean value of positions converges to the optimal point $p^* = -2$.

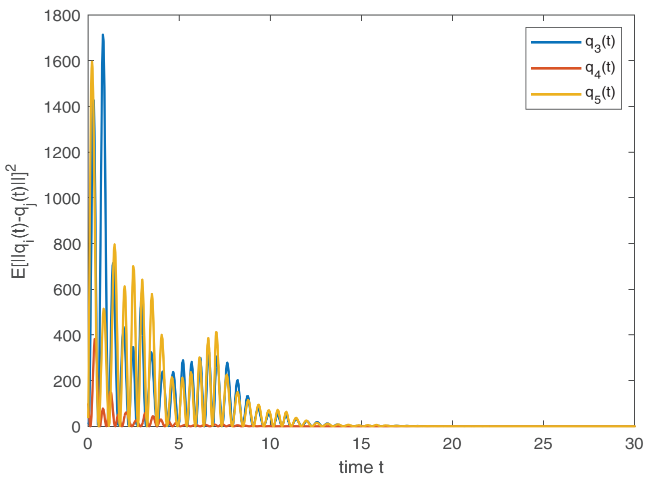

FIGURE 2 Mean value of agents' velocity error over G_1 with $d(t) = 0.02|\cos t|$ and $\tilde{\theta} = 0.3$

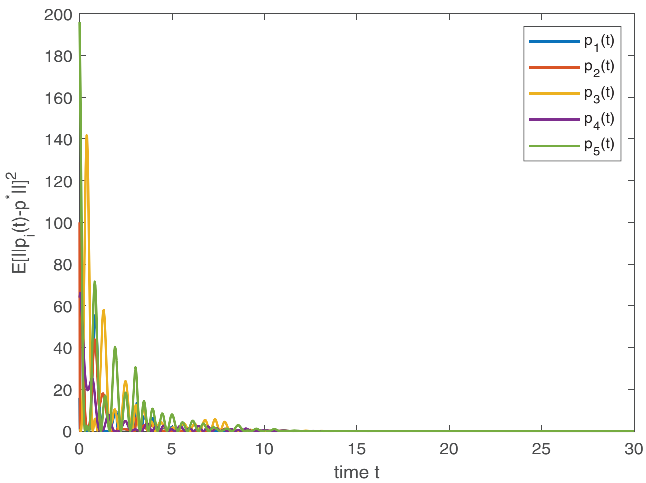

FIGURE 3 Mean value of agents' position error over switching topology with $d(t) = 0.02|\cos t|$ and $\tilde{\theta} = 0.3$

(ii) For systems (2.1) and (2.2) over the switching topology, which switches in sequence G_1, G_2, G_3 with the switching period $T = 1$ s,under the control protocols (3.24) and (3.25) with $\tilde{\theta} = 0.3$, $d(t) = 0.02|\cos t|$, the mean values of agents' state error are given in Figures 3 and 4, which illustrate that the mean value of all agents' states achieves together and the mean value of position states converges to the optimal point $p^* = -2$.

(iii) For systems (2.1) and (2.2) over G_1, under protocols (3.1) and (3.2), Figures 5 and 6 show the mean value of all agents' position and velocity error with $d(t) = 0.02|\cos t|$, $\tilde{\theta} = 0.6$. For systems (2.1) and (2.2) over G_1 under protocols (3.1) and (3.2), when $\tilde{\theta} = 0.9$, by solving the linear matrix inequality (3.14) one can get $d_0 \leq 0.017$. Figures 7 and 8 and Figures 11 and 12 show the mean value of all agents' position and velocity error with $d(t) = 0.015|\cos t|$,$\tilde{\theta} = 0.9$, and $d(t) = 0.02|\cos t|$, $\tilde{\theta} = 0.9$, respectively. Figures 7 and 8 illustrate that the mean value of positions for agents

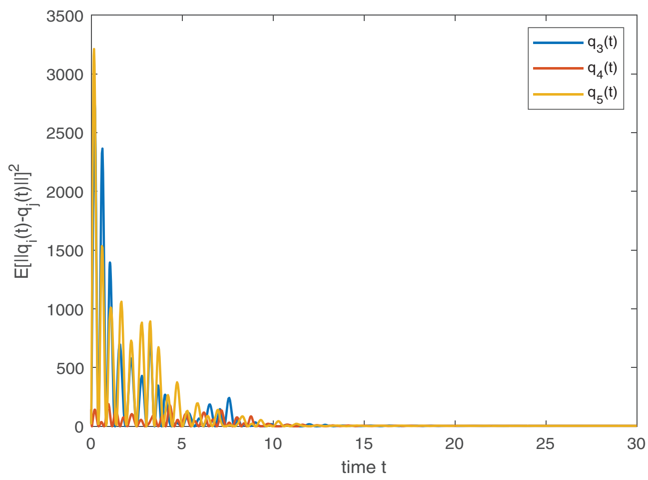

FIGURE 4 Mean value of agents' velocity error over switching topology with $d(t) = 0.02|\cos t|$ and $\tilde{\theta} = 0.3$

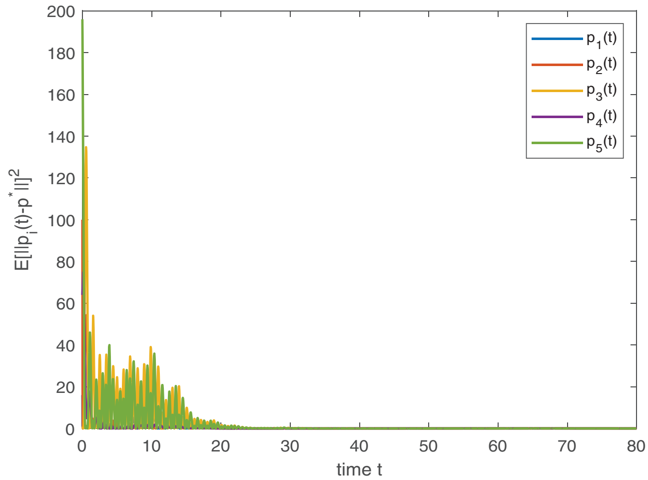

FIGURE 5 Mean value of agents' position error over G_1 with $d(t) = 0.02|\cos t|$ and $\tilde{\theta} = 0.6$

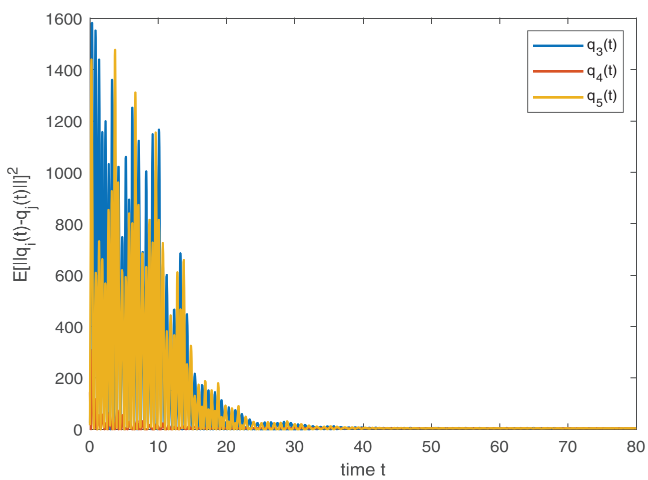

FIGURE 6 Mean value of agents' velocity error over G_1 with $d(t) = 0.02|\cos t|$ and $\tilde{\theta} = 0.6$

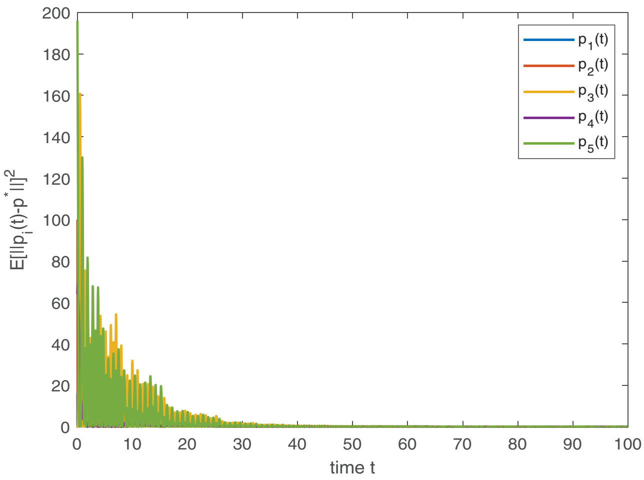

FIGURE 7 Mean value of agents' position error over G_1 with $d(t) = 0.015|\cos t|$ and $\tilde{\theta} = 0.9$

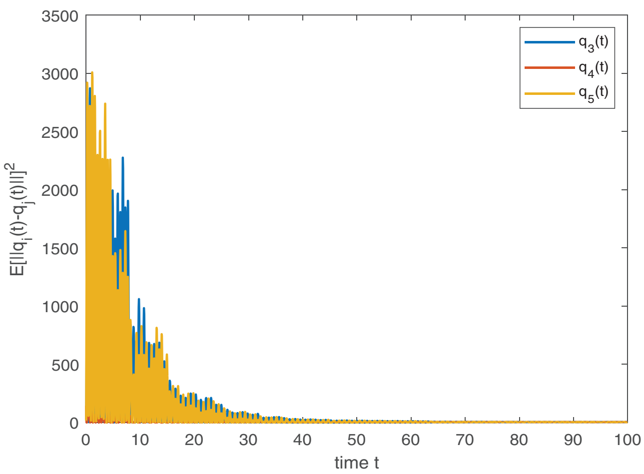

FIGURE 8 Mean value of agents' velocity error over G_1 with $d(t) = 0.015|\cos t|$ and $\tilde{\theta} = 0.9$

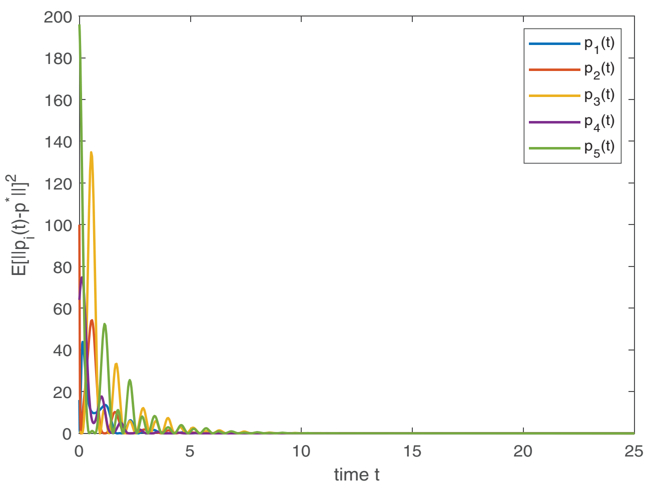

FIGURE 9 Mean value of agents' position error over G_1 with $d(t) = 100|\cos t|$ and $\tilde{\theta} = 0$

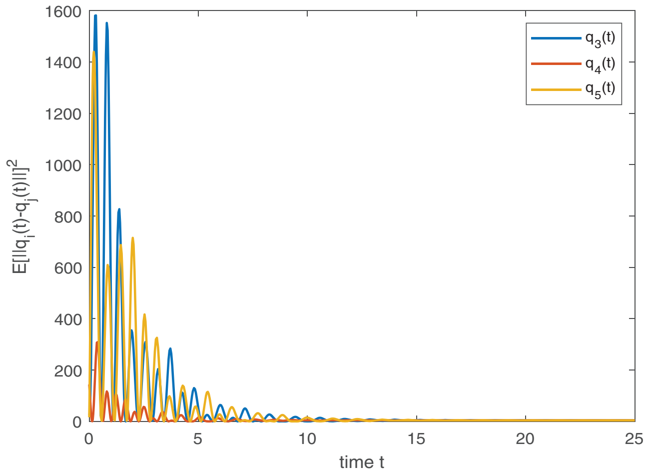

FIGURE 10 Mean value of agents' velocity error over G_1 with $d(t) = 100|\cos t|$ and $\tilde{\theta} = 0$

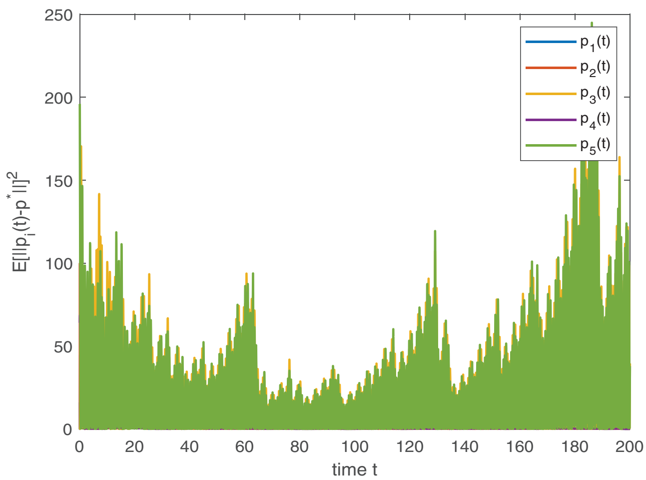

FIGURE 11 Mean value of agents' position error over G_1 with $d(t) = 0.02|\cos t|$ and $\tilde{\theta} = 0.9$

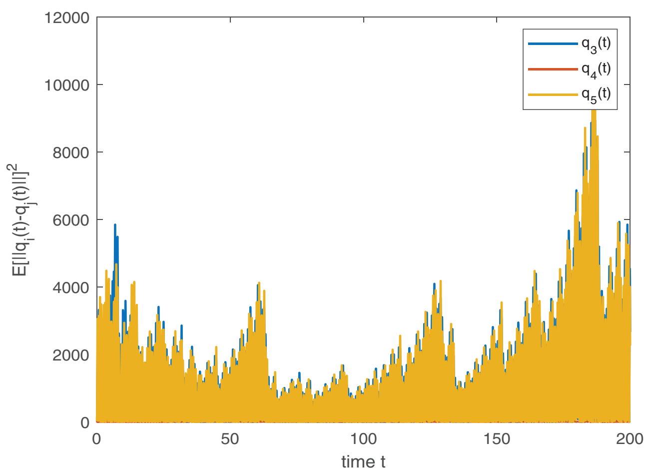

FIGURE 12 Mean value of agents' velocity error over G_1 with $d(t) = 0.02|\cos t|$ and $\tilde{\theta} = 0.9$

reach the optimal point $p^* = -2$ and all the velocities reach together, while Figures 11 and 12 illustrate that the mean value of agents' positions and velocities cannot achieve together when $d_0 > 0.017$.

(iv) For systems (2.1) and (2.2) over G_1, under protocols (3.1) and (3.2) with $d(t) = 100|\cos t|$ and $\tilde{\theta} = 0$, the mean value trajectories of agents' state error are given in Figures 9 and 10, which show that the mean value of all agents' states achieve together.

Remark 4.1. Figures 1 and 2 and Figures 5 and 6 illustrate that, the mean values of agents' states achieve together, and the consensus state makes the objective function $g(p) = \sum_{i=1}^{5} g_i(p)$ optimal, which verifies the effectiveness of the main results.

Remark 4.2. Comparing Figures 1 and 2 to Figures 7 and 8 and Figures 11 and 12 one can find that, the greater the probability of time delay, the slower the convergence velocity. Compared with Figures 1 and 2 and Figures 11 and 12 one can find, the greater the probability of time delay, the smaller the maximum tolerable time delay.

Remark 4.3. Figures 9 and 10 illustrate that if $\tilde{\theta} \equiv 0$, that is, the probability of time delay is zero, then the time delay does not affect the convergence.

5 | CONCLUSIONS

This paper studies the optimal mean-square consensus for HMAS both over fixed and switched weighted-balanced topologies. By adopting probability statistics, stochastic process, matrix theory and stability method, the control protocol is designed and sufficient conditions for the optimal consensus are obtained. The presented simulations verify the potential correctness of the main results. The related problem of systems with noises or stochastic network is the future work to be done.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

PERMISSION STATEMENT TO REPRODUCE THE MATERIALS FROM THE OTHER SOURCES

None.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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ORCID

Fenglan Sun https://orcid.org/0000-0003-1250-0566
Hao Li https://orcid.org/0000-0003-0092-5145
Jürgen Kurths https://orcid.org/0000-0002-5926-4276

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