DOI: https://doi.org/10.1063/5.0087839

Chaos
An Interdisciplinary Journal of Nonlinear Science

Oscillation quenching in diffusively coupled dynamical networks with inertial effects

Cite as: Chaos **32**, 041102 (2022); https://doi.org/10.1063/5.0087839
Submitted: 10 February 2022 • Accepted: 22 March 2022 • Published Online: 06 April 2022

Wei Zou, Yuxuan Chen, D. V. Senthilkumar, et al.

Oscillation quenching in diffusively coupled dynamical networks with inertial effects

Cite as: Chaos **32**, 041102 (2022); doi: 10.1063/5.0087839
Submitted: 10 February 2022 · Accepted: 22 March 2022 ·
Published Online: 6 April 2022

View Online Export Citation CrossMark

Wei Zou,[1,a)] Yuxuan Chen,[1] D. V. Senthilkumar,[2] and Jürgen Kurths[3,4,5]

AFFILIATIONS

[1]School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China
[2]School of Physics, Indian Institute of Science Education and Research, Thiruvananthapuram 695551, Kerala, India
[3]Potsdam Institute for Climate Impact Research, Telegraphenberg, Potsdam D-14415, Germany
[4]Institute of Physics, Humboldt University Berlin, Berlin D-12489, Germany
[5]World-Class Research Center "Digital Biodesign and Personalized Healthcare," Sechenov First Moscow State Medical University, Moscow 119991, Russia

[a)]**Author to whom correspondence should be addressed:** weizou83@gmail.com

ABSTRACT

Self-sustained oscillations are ubiquitous and of fundamental importance for a variety of physical and biological systems including neural networks, cardiac dynamics, and circadian rhythms. In this work, oscillation quenching in diffusively coupled dynamical networks including "inertial" effects is analyzed. By adding inertia to diffusively coupled first-order oscillatory systems, we uncover that even small inertia is capable of eradicating the onset of oscillation quenching. We consolidate the generality of inertia in eradicating oscillation quenching by extensively examining diverse quenching scenarios, where macroscopic oscillations are extremely deteriorated and even completely lost in the corresponding models without inertia. The presence of inertia serves as an additional scheme to eradicate the onset of oscillation quenching, which does not need to tailor the coupling functions. Our findings imply that inertia of a system is an enabler against oscillation quenching in coupled dynamical networks, which, in turn, is helpful for understanding the emergence of rhythmic behaviors in complex coupled systems with amplitude degree of freedom.

During the past decade, in order to revive the deprived rhythmic activities of coupled dynamical networks, various schemes have been proposed to eradicate the onset of *oscillation quenching*. However, most of the previously proposed techniques have their own limitations and drawbacks as they inevitably alter the intrinsic coupling functions, which generally cannot be simply modified in some real-world complex systems. Thus, it is meaningful to establish methods by which the onset of oscillation quenching can be eradicated without tailoring the coupling functions. In this paper, we analyze the onset of oscillation quenching in diffusively coupled dynamical networks including "inertial" effects. Interestingly, we unveil that the inclusion of (even a small) inertia is capable of eradicating the onset of oscillation quenching. To validate the generic role of inertia, we carefully examine distinctly different quenching scenarios in the paradigmatic models of coupled Stuart–Landau (SL) limit-cycle oscillators, including a pair of SL oscillators with delayed and dynamic couplings, a mean-field system of *N* globally coupled SL oscillators with randomly distributed frequencies, and an arbitrary network of delay-coupled SL oscillators, where quenching of oscillations has been previously reported to be the onset in the case of no inertia. Moreover, to deeply understand the essential role of inertia, we theoretically analyze a general form of diffusively coupled second-order dynamical systems with inertia in a fully general setting, which provides a common framework for understanding the role of inertia in eradicating the onset of oscillation quenching by destabilizing the quenched states.

I. INTRODUCTION

Coupled dynamical networks constitute an excellent framework for dissecting a plethora of collective behaviors that spontaneously emerge in many fields of science and engineering.[1–5] Quite recently, various phenomena have been discovered and explored in coupled oscillatory networks.[6,7] Among them, *oscillation quenching*, which leads to a cessation of macroscopic rhythmic behaviors of coupled oscillatory systems, has witnessed tremendous progress in recent years.[8–13] Quenching of oscillations in coupled dynamical networks generally takes place via two distinct manifestations: amplitude death and oscillation death.[9,10] Amplitude death occurs by stabilizing an unstable homogeneous steady state (HSS), whereas oscillation death manifests through a stable inhomogeneous steady state (IHSS) created by the coupling.

However, a rich of intriguing self-organized phenomena of complex real-world systems originate from the sustained evolutionary nature of the interacting oscillatory units. For many realistic networks, rhythmic activity is deemed to be a fundamental prerequisite to ensure their intrinsic behaviors[13] such as spiking neural networks, cardiac and respiratory systems, electric power generators, and numerous other natural instances.[14] A complete loss of macroscopic oscillations in these real-world systems is destructive and unwanted, thus the onset of oscillation quenching should be circumvented to mitigate possible fatal consequences. Devising strategies to efficiently eradicate the onset of oscillation quenching in coupled dynamical networks has been highlighted as a challenging as well as a practically significant issue,[8,13] whose studies are of importance to construct resilient dynamical networks. Recently, considerable attention has been paid to this topic;[13] however, all the existing techniques inevitably modify the coupling functions,[15–19] which could not be simply altered in some real situations. Thus, it is meaningful to establish additional methods by which oscillation quenching can be effectively weakened or even completed eradicated without tailoring the coupling functions.

To better describe real systems, inertia has been included to refine the phase models,[20–26] which features additional nontrivial and interesting dynamical consequences such as hysteretic behaviors or explosive synchronization. Incorporation of inertia is crucial for better understanding synchronous dynamics of fireflies,[27] coupled Josephson junctions,[28,29] power grids,[30,31] phase-locked loops,[32] an array of pendula,[33–37] etc. Hitherto, to our knowledge, even though the second-order Kuramoto phase model with inertia has been extensively studied,[20–26] the effects of inertia have not yet been adequately addressed in coupled Stuart–Landau (SL) limit-cycle systems, which involve both phase and amplitude dynamics.

In this work, we examine the onset of oscillation quenching in diffusively coupled SL limit-cycle systems including "inertial" effects. By adding inertia to diffusively coupled first-order oscillatory systems, we uncover that the presence of (even a tiny amount of) inertia can eradicate the onset of oscillation quenching. We establish the generic nature of inertia via carefully examining the paradigmatic systems of coupled SL oscillators under distinctly different scenarios for which oscillation quenching has been previously reported to occur in the absence of inertia. We further validate the destabilizing role of inertia in a fully general setting by considering a general form of coupled dynamical networks with inertia. The proposed scheme via inertia provides an additional way to eradicate oscillation quenching in coupled dynamical networks, which might be helpful for understanding the origin of rhythmicity of realistic complex systems comprised of a large ensemble of coupled dynamical units.

II. TWO COUPLED STUART–LANDAU OSCILLATORS WITH INERTIA

We start by considering a system of two coupled Stuart–Landau (SL) limit-cycle oscillators with inertia described by

$$m\ddot{Z}_j(t) + \dot{Z}_j(t) = (1 + i\omega_j - |Z_j(t)|^2)Z_j(t) + K[Z_k(t-\tau) - Z_j(t)], \tag{1}$$

where $j, k = 1, 2$ $(j \neq k)$, $Z_j(t)$ and ω_j are the complex amplitude and the natural rotational frequency of the jth SL oscillator, K mediates the strength of coupling, and τ quantifies the propagation delay. The first term $m\ddot{Z}_j(t)$ $(m \geq 0)$ on the left-hand side of Eq. (1) can be interpreted as *inertia* of the jth element. Note that inertia has been previously employed to generalize the paradigmatic Kuramoto phase model,[38] which, however, has not still been thoroughly considered in the models that incorporate both the phase and amplitude dynamics. For $m = 0$ and $K = 0$, each uncoupled SL oscillator exhibits a stable limit-cycle motion $Z(t) = e^{i\omega_j t}$ and has an unstable focus at $Z(t) = 0$. The SL oscillator is a canonical model describing the dynamics near a supercritical Hopf bifurcation, which has been widely used to explore diverse collective behaviors (in particular, amplitude death and oscillation death) of coupled dynamical networks for decades.

For $K > 0$, the transition from oscillatory to quiescent states, i.e., quenching of oscillations via amplitude death, in system (1) has been well examined by Aronson *et al.* for $m = \tau = 0$[39] and by Reddy *et al.* for $m = 0$ and $\tau > 0$.[40,41] They reported that quenching of oscillations (amplitude death) occurs in two coupled SL oscillators (1) via stabilizing the HSS $Z_1 = Z_2 = 0$, where the HSS is proved to be stable within a large interval of K for sufficiently disparate frequencies when $\tau = 0$[39] or even for identical frequencies if $\tau > 0$.[40,41] Here, we will first unveil that the quenching of oscillations in (1) can be eradicated by finite inertia with $m > 0$ via showing that even tiny inertia successfully destabilizes the stable HSS under the above two scenarios.

When the coupled system (1) with $m = 0$ experiences the oscillation quenching, the limit-cycle oscillations of all individuals completely lose their stability due to the coupling, and meanwhile the origin $Z_1(t) = Z_2(t) = 0$ becomes stabilized. By performing a standard linear stability analysis of (1) around $Z_1(t) = Z_2(t) = 0$, the condition for the onset of the stable HSS is determined from the characteristic equation

$$\prod_{j=1}^{2}(m\lambda^2 + \lambda - 1 - i\omega_j + K) - K^2 e^{-2\lambda\tau} = 0. \tag{2}$$

The stability of the HSS is switched when the rightmost eigenvalue (i.e., the eigenvalue with the largest real part) of (2) crosses the imaginary axis in the complex plane.

For $\tau = 0$, Eq. (2) is reduced to

$$m\lambda^2 + \lambda + K - 1 - i\bar{\omega} \pm \frac{1}{2}\sqrt{4K^2 - \Delta^2} = 0, \tag{3}$$

with $\overline{\omega} = (\omega_1 + \omega_2)/2$ and $\Delta = |\omega_1 - \omega_2|$. Figure 1(a) plots the rightmost eigenvalue of (3) as a function of m, where $K = 2, \overline{\omega} = 2$, and $\Delta = 5$ are fixed. For increasing m from zero, it can be observed that the rightmost eigenvalue of (2) crosses the imaginary axis from

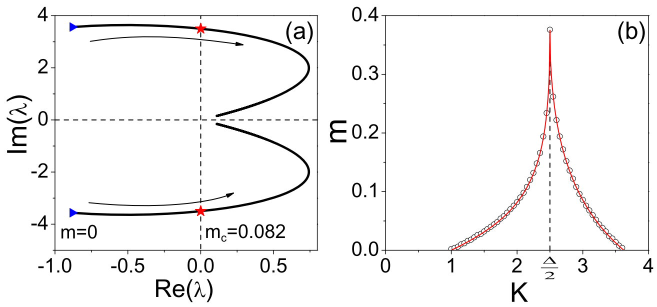

FIG. 1. Eradicating the onset of oscillation quenching by inertia in two coupled SL oscillators (1) with $\tau = 0$ and $\omega_1 \neq \omega_2$. In this scenario, for $m = 0$, amplitude death takes place due to the interplay of a sufficient mismatch between two intrinsic frequencies and a large coupling strength. (a) The rightmost eigenvalue of (3) as a function of m. $K = 2$ is used. Locations of the rightmost eigenvalues for $m = 0$ and m_c are highlighted by blue triangle and red star, respectively. The arrows indicate the direction of increasing m. (b) Stable HSS interval vs m. $\omega_1 = \overline{\omega} - \Delta/2$ and $\omega_2 = \overline{\omega} + \Delta/2$ with $\overline{\omega} = 2$ and $\Delta = 5$ are fixed for both panels.

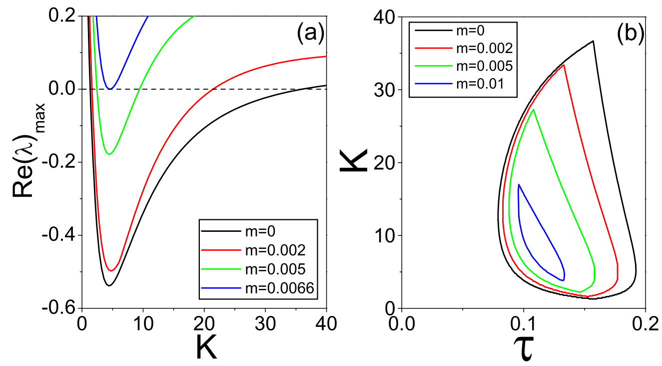

FIG. 2. Eradicating the onset of oscillation quenching by inertia in two coupled SL oscillators (1) with $\tau > 0$ and $\omega_1 = \omega_2 = \omega$. In this situation, for $m = 0$, quenching of oscillations arises because of the presence of time delay in the coupling. (a) The largest real part $\mathrm{Re}(\lambda)_{\max}$ of Eq. (5) vs K for different values of m. $\tau = 0.15$ and $\omega = 10$ are used. (b) Stable region of the HSS in the parameter space of (τ, K) as a function of m.

the left to the right at $m_c = 0.082$. For $m > m_c$, the stability of the stable HSS is reversed, thereby indicating that oscillation quenching in the case of $m = 0$ is successfully eradicated by inertia.

For $m = 0$ in (3), the HSS is stable for $1 < K < (1 + \Delta^2/4)/2$ if $\Delta > 2$.[39] By setting $\mathrm{Re}(\lambda) = 0$ in (3) and separating the real and imaginary parts, we derive critical inertia

$$m_c(K) = \begin{cases} \frac{K-1}{\left(|\overline{\omega}| + \frac{1}{2}\sqrt{\Delta^2 - 4K^2}\right)^2} & \text{if } 1 < K \leqslant \frac{\Delta}{2}, \\ \frac{K - 1 - \frac{1}{2}\sqrt{4K^2 - \Delta^2}}{\overline{\omega}^2} & \text{if } \frac{\Delta}{2} < K \leqslant \frac{1}{2} + \frac{\Delta^2}{8} \end{cases} \tag{4}$$

beyond which the stable HSS becomes destabilized for a given K. At $K = \Delta/2$, m_c reaches its maximum $m_{max} = (\Delta/2 - 1)/\overline{\omega}^2$. For instance, Fig. 1(b) depicts the stable HSS interval as a function of m, where $\overline{\omega} = 2$ and $\Delta = 5$ are used as in Fig. 1(a). The theoretical prediction in Eq. (4) (red lines) is in excellent agreement with numerical simulations (open circles) of coupled system (1). Clearly, the stable HSS interval monotonically decreases as m increases, and no longer exists for $m > m_{max} = 0.375$, which implies that the HSS is impossible to be stabilized for any $K > 0$, thus oscillation quenching is eradicated.

For $\tau > 0$, Reddy *et al.*[40,41] reported the onset of the stable HSS in (1) with $m = 0$ even for $\omega_1 = \omega_2 = \omega$. Now, we will probe the effect of m in this case of *death by delay*.[42] For $\Delta = 0$, Eq. (2) is further reduced to

$$m\lambda^2 + \lambda = 1 + i\omega - K \pm K e^{\lambda\tau}. \tag{5}$$

The HSS is stable if all the roots of (5) are in the open left half of the complex plane. Figure 2(a) plots the largest real part $\mathrm{Re}(\lambda)_{\max}$ of (5) vs K for $m = 0$, 0.002, 0.005, and 0.0066, respectively, where $\tau = 0.15$ and $\omega = 10$ are fixed. Evidently, increasing m results in a monotonic shrinkage of the stable interval of the HSS, which vanishes for $m > m_c = 0.066$. For a global picture, Fig. 2(b) further plots the stable HSS region in the (τ, K) space for $m = 0$, 0.002, 0.005, and 0.01, respectively. Clearly, the stable HSS region monotonically shrinks as successively enhancing the level of system's inertia, which completely disappears in the whole parameter space of (τ, K) once $m > m_c \approx 0.014$. Thus, for all the time delay, the presence of inertia is able to eradiate the onset of oscillation quenching in two coupled SL oscillators (1). In this sense, the stability results can be said to be valid for any delay.

To elucidate the generic nature of inertia, we will demonstrate that it is also capable of eradicating the onset of oscillation quenching by destabilizing the HSS in two SL oscillators with dynamic coupling[43]

$$\begin{aligned} m\ddot{Z}_j + \dot{Z}_j &= (1 + i\omega - |Z_j|^2)Z_j + K(u_j - Z_j), \\ \dot{u}_j &= -u_j + Z_k, \end{aligned} \tag{6}$$

where $j, k = 1, 2$ $(j \neq k)$. A linear stability analysis of (6) around $Z_j = 0$ results in the characteristic equation

$$m\lambda^2(\lambda + 1) + \lambda^2 + (K - i\omega)\lambda - 1 - i\omega + K \pm K = 0. \tag{7}$$

For $m = 0$, the stable interval of the HSS is given by $(\omega^2 - \omega\sqrt{\omega^2 - 4})/2 < K < (\omega^2 + \omega\sqrt{\omega^2 - 4})/2$ if $\omega > 2$. The largest real part $\mathrm{Re}(\lambda)_{\max}$ of Eq. (7) with $\omega = 3$ as a function of K for $m = 0$, 0.1, 0.2, and 0.3 is depicted in Fig. 3(a). It is evident that an increase in m leads to a monotonous decrease in the stable interval of the HSS. For a global perspective, Fig. 3(b) further delineates the stable region of the HSS in the parameter (K, ω) space, which elucidates that the spread of the stable HSS region shrinks on increasing m and eventually vanishes for $m > m_c$, thereby corroborating the generic effect of inertia in eradicating the occurrence of oscillation quenching in two coupled SL oscillators.

In our study, we have further confirmed the generic nature of inertia in eradicating amplitude death in diverse systems of two SL oscillators with many other coupling configurations, such as distributed-delay coupling,[44] conjugate coupling,[45] mean-field diffusion,[46] etc. Note that the presence of inertia, i.e., $m > 0$, does not deform the structure of the steady states of coupled dynamical

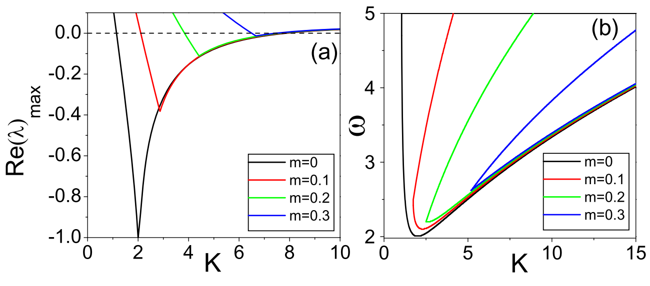

FIG. 3. Eradicating the onset of oscillation quenching by inertia in two coupled SL oscillators with dynamic coupling (6). (a) The largest real part $Re(\lambda)_{max}$ of Eq. (7) with $\omega = 3$ vs K for $m = 0$, 0.1, 0.2, and 0.3. (b) Stable HSS regions in the (K, ω) parameter plane.

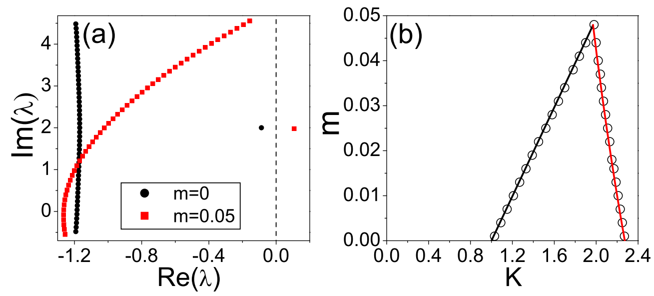

FIG. 4. Eradicating the onset of oscillation quenching by inertia in a heterogeneous population of globally coupled SL oscillators (8). (a) Distribution of eigenvalues in (9) for $m = 0$ (black circles) and $m = 0.05$ (red squares). (b) Stable coupling interval of the HSS as a function of m. Black and red lines denote the theoretical predictions of (13) and (14) with $\bar{\omega} = 2$ and $\Delta = 5$, respectively, where open circles represent the numerical results.

systems but which offers a possibility to modulate their stability. We have also consolidated the validity of inertia in destabilizing the stable IHSS (oscillation death) in two diffusively coupled Brusselators.[47] All the results corroborate that inertia can be generically employed to eradicate the onset of oscillation quenching in two coupled dynamical systems.

III. GLOBALLY COUPLED STUART–LANDAU OSCILLATORS WITH INERTIA

The virtue of inertia in eradicating the onset of oscillation quenching is not restricted to two coupled oscillators, which is valid even in a large population of coupled oscillators. Now, let us study a system of globally and diffusively coupled SL oscillators with inertia represented as

$$m\ddot{Z}_j + \dot{Z}_j = (1 + i\omega_j - |Z_j|^2)Z_j + \frac{K}{N}\sum_{k=1}^{N}(Z_k - Z_j), \tag{8}$$

where $j = 1, 2, \ldots, N\,(N > 2)$, ω_j's are randomly distributed according to a prescribed density function $g(\omega)$. In the limit of a weak coupling $K \to 0$, Eq. (8) can be reduced to the standard Kuramoto model with inertia.[20,21] Note that Eq. (8) is recovered to the same model as in Refs. 48 and 49 with $m = 0$, where the loss of macroscopic oscillations due to a stabilization of the HSS (i.e., the onset of amplitude death) has been reported for a sufficiently broad distribution of natural frequencies.[50–52] Next, we will show that the presence of inertia $m > 0$ can indeed reverse the stability of the stable HSS in coupled system (8).

After carrying out a standard linear stability analysis of (8) around $Z_j = 0$, the onset of the stable HSS can be identified from the characteristic equation

$$\prod_{j=1}^{N}(\mu - iw_j)\left(1 - \frac{K}{N}\sum_{j=1}^{N}\frac{1}{\mu - iw_j}\right) = 0, \tag{9}$$

with $\mu = m\lambda^2 + \lambda + K - 1$. The HSS is stable if all the roots of (9) have negative real parts. Figure 4(a) shows two typical distributions of characteristic eigenvalues in (9) with $m = 0$ (black circles) and $m = 0.05$ (red squares), where $N = 50$, $K = 2.2$, and the frequencies ω_j's are uniformly sampled on $[-0.5, 4.5]$. Note that the real part of the rightmost eigenvalue shifts from negative ($m = 0$) to positive ($m = 0.05$), which signs a successful destabilization of the stable HSS caused by increasing m. In the thermodynamic limit, $N \to \infty$, the roots of (9) can be analytically tractable, which are composed of the continuous and discrete spectra given by

$$m\lambda^2 + \lambda + K - 1 - i\omega = 0 \tag{10}$$

and

$$\frac{1}{K} = \int_{-\infty}^{+\infty}\frac{g(\omega)d\omega}{m\lambda^2 + \lambda + K - 1 - i\omega}, \tag{11}$$

respectively. Equations (10) and (11) provide a fairly accurate prediction of the onset of the stable HSS in the coupled system (8) with a large but finite N. By assuming ω_j to be uniformly distributed on $[\bar{\omega} - \Delta/2, \bar{\omega} + \Delta/2]$, the discrete spectrum in (11) can be further integrated to yield

$$m\lambda^2 + \lambda + K - 1 - i\bar{\omega} - \frac{\Delta}{2}\cot\left(\frac{\Delta}{2K}\right) = 0. \tag{12}$$

From Eqs. (10) and (12), two critical curves, enclosing the stable HSS, can be derived as

$$m_{c1}(K) = \frac{K - 1}{\left(\bar{\omega} + \frac{\Delta}{2}\right)^2} \tag{13}$$

and

$$m_{c2}(K) = \frac{K - 1 - \frac{\Delta}{2}\cot\left(\frac{\Delta}{2K}\right)}{\bar{\omega}^2}. \tag{14}$$

Figure 4(b) plots the theoretical predictions of (13) (black line) and (14) (red line) with $\bar{\omega} = 2$ and $\Delta = 5$, which have been fully supported by the numerical results (open circles) of the coupled system (8) with $N = 200$. Clearly, quenching of oscillations is impossible to occur for any $K > 0$ when $m > m_c \approx 0.048$, which implies the ability of inertia in eradicating the onset of oscillation quenching in heterogeneous populations of globally coupled SL oscillators (8).

IV. NETWORKS OF STUART–LANDAU OSCILLATORS WITH INERTIA

Next, we demonstrate that the effect of inertia persists even for networks of delay-coupled SL oscillators. Specifically, we consider networked delay-coupled SL oscillators with inertia represented as

$$m\ddot{Z}_j + \dot{Z}_j = (1 + i\omega - |Z_j|^2)Z_j + \frac{K}{d_j}\sum_{k=1}^{N} g_{jk}[Z_k(t-\tau) - Z_j(t)], \tag{15}$$

where $j = 1, \ldots, N\,(N \geq 2)$, g_{jk} encodes the topology of the underlying network, that is, if jth and kth nodes are connected by a link, then $g_{jk} = g_{kj} = 1$, otherwise $g_{jk} = g_{kj} = 0$, $g_{jj} = 0$. $d_j = \sum_{k=1}^{N} g_{jk}$ represents the degree of the jth node. From a linear stability analysis of (15) around $Z_j = 0$, the characteristic equations governing the stability of the HSS can be deduced as

$$m\lambda^2 + \lambda = 1 + iw - K + K\rho_j e^{-\lambda\tau}, \tag{16}$$

with ρ_j's $(j = 1, 2, \ldots, N)$ denoting the eigenvalues of the matrix $G = \left(\frac{g_{js}}{d_j}\right)_{N\times N}$, which can be ordered as $1.0 = \rho_1 \geq \rho_2 \geq \cdots \geq -\frac{1}{N-1} \geq \rho_N \geq -1.0$.[53] The HSS is stable if $\mathrm{Re}(\lambda)_{\max}$ of (16) for each ρ_j is negative.

Figure 5(a) plots a typical dependence of $\mathrm{Re}(\lambda)_{\max}$ of (16) on ρ_j, from which one can find that the bounding modes are the first ones to become destabilized, thus revealing that the stable region of the HSS is defined only by the largest and smallest eigenvalues: ρ_1 and ρ_N. As $\rho_1 = 1$ is fixed, the impact of the coupling topology on the HSS is solely captured by ρ_N $(0 \geq \rho_N \geq -1.0)$; and the larger ρ_N, the larger the stable HSS region is. Figure 5(b) depicts the stable interval of the HSS vs ρ_N for $m = 0, 0.01, 0.02$, and 0.04, respectively, where $\omega = 10$ and $\tau = 0.2$ are fixed. For each ρ_N, the stable interval of the HSS strongly reduces for increasing m and disappears when m is beyond a critical threshold. Thus, inertia is able to eradicate the onset of oscillation quenching in the delay-coupled networks (15).

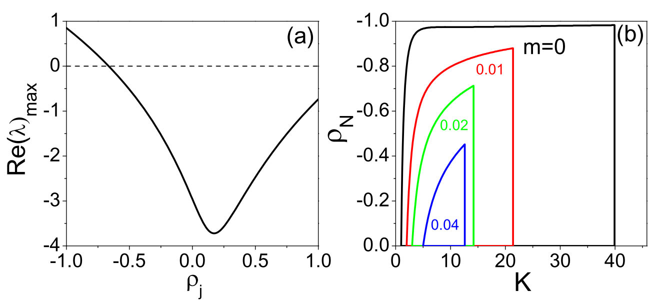

FIG. 5. Eradicating the onset of oscillation quenching by inertia in networked delay-coupled SL oscillators (15). For $m = 0$, quenching of oscillations in (15) can be induced by the presence of time delay in the coupling. (a) A typical dependence of the largest real part $\mathrm{Re}(\lambda)_{\max}$ of (16) on ρ_j, where $m = 0.01$, $\omega = 10$, $K = 5$, and $\tau = 0.2$. (b) Stable coupling interval of the HSS of (15) vs ρ_N with $m = 0, 0.01, 0.02$, and 0.04. The role of topology of the coupled networks is solely characterized by the value of ρ_N.

V. GENERAL FORMS OF COUPLED DYNAMICAL NETWORKS WITH INERTIA

In order to acquire a more general understanding of the nature of inertia in eradicating the onset of oscillation quenching, let us finally consider the following general form of coupled dynamical networks with inertia defined by

$$m\ddot{\mathbf{X}}_j + \dot{\mathbf{X}}_j = \mathbf{F}(\mathbf{X}_j, \mu_j) + K\sum_{k=1}^{N} \mathbf{L}_{jk}\mathbf{H}(\mathbf{X}_k), \tag{17}$$

where $j = 1, \ldots, N\,(N \geq 2)$, the variable $\mathbf{X}_j \in \mathbf{R}^n$ and $\mu_j \in \mathbf{R}$ denote, respectively, a n-dimensional state vector and an intrinsic parameter of the jth system, $\mathbf{H}: \mathbf{R}^n \to \mathbf{R}^n$ is the coupling function, and the $N \times N$ (adjacency or Laplacian) matrix $\mathbf{L}$ codes the network topology.[54,55] The dynamics of the jth node is governed by $\dot{\mathbf{X}}_j = \mathbf{F}(\mathbf{X}_j, \mu_j)$ with the function $\mathbf{F}: \mathbf{R}^n \to \mathbf{R}^n$, which is assumed to exhibit a non-stationary behavior such as periodic (or even chaotic) oscillations. Note that Eq. (17) can be considered as the standard form of second-order mechanical systems, where each X_j denotes a degree of freedom in the system.[56] Suppose that, quenching of oscillations takes place in the coupled networks (17) with $m = 0$ for a certain value of $K > 0$, i.e., macroscopic oscillations of the coupled networks are quenched by the coupling via stabilizing a steady state $\mathbf{X}^* = [\mathbf{X}_1^{*T}, \mathbf{X}_2^{*T}, \ldots, \mathbf{X}_N^{*T}]^T$, which can be either a HSS or an IHSS. The presence of inertia with $m > 0$ dose not change the steady-state solution $\mathbf{X}^*$ of (17), which, however, may exert a great impact on its stability. In the following, we will prove that the onset of oscillation quenching in (17) can be eradicated by inertia, via showing that the stability of $\mathbf{X}^*$ can be completely reversed once m is increased beyond a critical threshold m_c.

Linearizing (17) around $\mathbf{X}_j^*$, we have the variational equation

$$m\delta\ddot{\mathbf{X}}_j + \delta\dot{\mathbf{X}}_j = J\mathbf{F}(\mathbf{X}_j^*, \mu_j)\delta\mathbf{X}_j + K\sum_{k=1}^{N} \mathbf{L}_{jk}J\mathbf{H}(\mathbf{X}_k^*)\delta\mathbf{X}_k, \tag{18}$$

where $\delta\mathbf{X}_j$ represents a small perturbation of the jth node from $\mathbf{X}_j^*$ and J is the Jacobian operator. The linear system (18) can be described by

$$m\delta\ddot{\mathbf{X}} + \delta\dot{\mathbf{X}} = \mathbf{A}\delta\mathbf{X}, \tag{19}$$

with $\delta\mathbf{X} = [\delta\mathbf{X}_1^T, \delta\mathbf{X}_2^T, \ldots, \delta\mathbf{X}_N^T]^T$. For $m = 0$, quenching of oscillations is supposed to occur as a consequence of the coupling in (17) via stabilizing $\mathbf{X}^*$, thus the rightmost characteristic eigenvalue, $a \pm ib$, of the matrix $\mathbf{A}$ has a negative real part, i.e., $a < 0$ $(b \geq 0)$. By letting $\delta\mathbf{Y} = \delta\dot{\mathbf{X}}$, Eq. (19) can be written as

$$\begin{pmatrix}\delta\dot{\mathbf{X}}\\ \delta\dot{\mathbf{Y}}\end{pmatrix} = \begin{pmatrix}0 & \mathbf{I}_N\\ \frac{1}{m}\mathbf{A} & -\frac{1}{m}\mathbf{I}_N\end{pmatrix}\begin{pmatrix}\delta\mathbf{X}\\ \delta\mathbf{Y}\end{pmatrix} \tag{20}$$

for $m > 0$, where $\mathbf{I}_N$ denotes the N-dimensional identity matrix.

The characteristic equation of the linear system (20) is defined by

$$\det\begin{pmatrix}\lambda\mathbf{I}_N & -\mathbf{I}_N\\ -\frac{1}{m}\mathbf{A} & \left(\lambda + \frac{1}{m}\right)\mathbf{I}_N\end{pmatrix} = 0. \tag{21}$$

After some matrix operations, Eq. (21) is reduced to

$$\det\left((m\lambda^2 + \lambda)\mathbf{I}_N - \mathbf{A}\right) = 0, \tag{22}$$

whose eigenvalue with the largest real part can be calculated from

$$m\lambda^2 + \lambda = a \pm ib \tag{23}$$

as

$$\lambda = \frac{-1 \pm \sqrt{1 + 4m(a \pm ib)}}{2m}. \tag{24}$$

Separating the real and imaginary parts of (23) with $\mathrm{Re}(\lambda) = 0$, critical inertia m_c is explicitly obtained as

$$m_c = -\frac{a}{b^2} \quad \text{for } b > 0, \tag{25}$$

which is in accordance with the results in Eqs. (4), (13), and (14). Differentiating (23) with respect to m, it is straightforward to get

$$\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}m}\bigg|_{m=m_c} = \frac{b^4}{4a^2 + b^2} > 0 \quad \text{if } b > 0, \tag{26}$$

which means that the rightmost eigenvalue λ will transit into the right half plane when increasing m beyond m_c.

Figures 6(a) and 6(b) schematically show the dependencies of the rightmost eigenvalue λ of the linear system (19) on m for $b > 0$ and $b = 0$, respectively. For $b > 0$, the eigenvalue crosses the imaginary axis from the left to the right at $m_c = -a/b^2$, whose real part asymptotically approaches to 0^+ as $m \to +\infty$. For $b = 0$, the eigenvalue first decreases from a to $2a$ as increasing m from 0 to $-1/4a$, and then its real part monotonically increases for $m > -1/4a$, which asymptotically approaches to 0^- as $m \to +\infty$. Thus, for a particularly special case of $b = 0$, the steady state $\mathbf{X}^*$ cannot be destabilized by m, but its stability can be strongly weakened if m is sufficiently large, which makes the coupled networks taking a very long transient time to achieve the quenched state. In contrast, for much more general situations with $b > 0$, there always exists a critical value of m_c, beyond which the stability of $\mathbf{X}^*$ can be completely reversed, therefore indicating that the deprived oscillations of (17) are successfully revived for all $m > m_c$. In this sense, the effect of inertia in eradicating the onset of oscillation quenching from quenched states can be said to be valid widely in coupled dynamical networks.

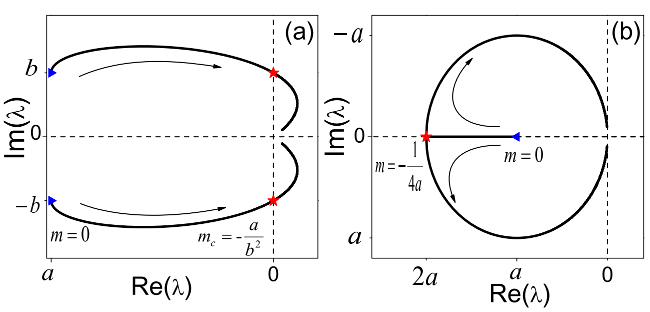

FIG. 6. Mechanisms of inertia in eradicating the onset of oscillation quenching in coupled dynamical networks of general form (17). (a) and (b) Schematics of the dependence of the rightmost eigenvalue λ in (24) on m for $b > 0$ and $b = 0$, respectively. The direction of increasing m is indicated by the arrows. For $m = 0$, quenching of oscillations is supposed to occur in (17) via stabilizing a steady state $\mathbf{X}^*$ for a certain coupling strength $K > 0$. The eigenvalue with the largest real part of the matrix $\mathbf{A}$, obtained by linearizing (17) with $m = 0$ around $\mathbf{X}^*$, is assumed to be $a \pm ib$ with $a < 0$ and $b \geq 0$.

VI. CONCLUSION AND DISCUSSIONS

By adding inertia to diffusively coupled first-order oscillatory systems, we found that the presence of even tiny inertia is capable of eradicating the onset of oscillation quenching, which takes place in the corresponding model with no inertia. To establish the generic nature of inertia, we have carried out the study progressively via carefully examining distinctly different quenching scenarios in the paradigmatic system of coupled SL limit-cycle oscillators, including a pair of SL oscillators with both delayed and dynamic couplings, a heterogeneous population of N globally coupled SL oscillators, and an arbitrary network of delay-coupled SL oscillators. Furthermore, a general form of coupled dynamical networks with inertia has been theoretically analyzed in a fully general setting, which provides a common framework for a systematic understanding of the essential role of inertia in eradicating the onset of oscillation quenching. In the main text, we have introduced the inertial terms in all the dynamical variables for deducing the stability conditions analytically; however, in our numerical simulations, we have validated that by associating inertia in only one of the variables of the n-dimensional dynamical system is sufficient to eradicate oscillation quenching via destabilizing the quenched states.

In terms of stability principle, the presence of inertia has the same type of effect on multiple dynamical systems studied, which generally makes the rightmost eigenvalue of the corresponding linearized systems crossing the imaginary axis from the left to the right in the complex plane. From the physical point of view, the common source of inertia in eradicating the phenomenon of oscillation quenching may be due to the fact that the additional inertial term actively pumps up additional energy into coupled dynamical networks. This effectively compensates the dissipation induced by the coupling, thus leading to a destabilization of quenched states.

It should be noted that all the theoretical results are obtained by performing standard linear stability analyses of the coupled system around the steady-state solution. Consequently, all the stability results are only valid in a neighborhood of the steady state around which the coupled system was linearized. The only thing that can be concluded from the linear stability analysis of coupled system around a fixed point is that the HSS (or IHSS) will be stable or unstable for certain intervals of system's parameters. The instability of the HSS, by no means, ensures the appearance of self-sustained oscillations. When the HSS becomes unstable, it is possible that the trajectories of the dynamical system converge to an IHSS. However, for all the quenching scenarios in the models of coupled SL oscillators considered in this study, we found that the destabilization of the stable HSS induced by inertia occurs via a supercritical Hopf bifurcation, which generally implies the emergence of limit-cycle oscillations. In our numerical simulations of coupled SL systems with inertia, we indeed observed that the limit-cycle oscillations are born after the HSS is destabilized by a certain amount of inertia. However, further rigorous analysis is required to determine that when the HSS becomes unstable, whether the only stable solution is oscillatory, which surely deserves a future study.

From the mathematical viewpoint, inertia is taken into account to make the dynamical system under consideration to be second-order ODEs, which tends to induce nontrivial behaviors even in a single uncoupled system such as promoting oscillations, bi-stability, or hysteresis.[2] In particular, the impact of inertia on synchronous dynamics of coupled phase-only systems (from the Kuramoto model to coupled pendula) has been an active research topic for a long time.[20–38] One of the surprising findings is that a first-order phase transition (hysteretic synchrony) can be resulted from a finite inertia in the Kuramoto model, which substantially differs from the no inertial case associated with a second-order transition to synchronization.[20–26] In the weak coupling limit, our model of globally coupled SL oscillators with inertia in Eq. (8) can be in fact reduced to the second-order Kuramoto phase model with inertia.[20,21] However, the dynamical roles of inertia in strongly coupled systems with both phase and amplitude dynamics have not been thoroughly explored before. Our current work has initiated a step in addressing this issue by revealing that the presence of inertia is detrimental to the onset of oscillation quenching in coupled dynamical networks.

By adding inertia, the original first-order dynamical system is transformed to a second-order one. Then, it is expected that the dynamic behavior of the transformed dynamical system will be, in general, different from that observed in the original one. We would like to emphasize that our results clearly indicate that the occurrence of oscillation quenching is less likely when a first-order system is translated to a second-order system by adding inertia. To clarify the point, we have deliberately conducted the study in the way by introducing inertia in the first-order SL limit-cycle system, which is due to the fact that the first-order SL oscillator model without inertia has been widely employed to explore the mechanisms of oscillation quenching.[8,13] In the future, from the very beginning, we should go beyond oscillation quenching to directly examine other collective behaviors in diffusively coupled second-order systems with inertia [e.g., Eq. (8)] such as the incoherent state, partially (fully) locked states, unsteady (periodic, quasiperiodic, and chaotic) motions,[48,49] chimeras,[57] exotic states,[58] etc., which will yield a more full picture of the overall dynamics of diffusively coupled second-order systems with inertia.

On the other side, we anticipate that the role of inertia in eradicating the onset of oscillation quenching could be evidenced in pertinent experiments such as coupled nonlinear electronic circuits,[59,60] electrochemical reactions,[61] thermoacoustic systems,[62] etc., where the phenomenon of oscillation quenching has already been observed. However, there may exist some possible challenges in the experimental verifications of our proposed solution such as how to obtain the inertial term and add inertia to the original first-order system. Finally, we believe that our present study will inspire a myriad of further research in coupled second-order systems with inertia, where both phase and amplitude degrees of freedom are involved.

ACKNOWLEDGMENTS

The authors would like to appreciate the two anonymous reviewers for their very thoughtful and constructive comments that helped them to greatly improve this paper. W.Z. acknowledges support from the National Natural Science Foundation of China (NNSFC, Grant No. 12075089) and the Natural Science Foundation of Guangdong Province, China (Grant No. 2019A1515011868). D.V.S. is supported by the DST-SERB-CRG Project under Grant No. CRG/2021/000816. J.K. is financed by the Ministry of Science and Higher Education of the Russian Federation within the framework of state support for the creation and development of World-Class Research Centers "Digital biodesign and personalized healthcare" (Grant No. 075-15-2020-926).

AUTHOR DECLARATIONS

Conflict of Interest

The authors declared that they have no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

REFERENCES

[1] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984).
[2] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2001).
[3] T. Stankovski, T. Pereira, P. V. E. McClintock, and A. Stefanovska, "Coupling functions: Universal insights into dynamical interaction mechanisms," Rev. Mod. Phys. **89**, 045001 (2017).
[4] P. S. Skardal and A. Arenas, "Control of coupled oscillator networks with application to microgrid technologies," Sci. Adv. **1**, e1500339 (2015).
[5] K. M. Hannay, D. B. Forger, and V. Booth, "Macroscopic models for networks of coupled biological oscillators," Sci. Adv. **4**, e1701047 (2018).
[6] X. Dai, X. Li, H. Guo, D. Jia, M. Perc, P. Manshour, Z. Wang, and S. Boccaletti, "Discontinuous transitions and rhythmic states in the D-dimensional Kuramoto model induced by a positive feedback with the global order parameter," Phys. Rev. Lett. **125**, 194101 (2020).
[7] C. Xu, X. H. Tang, H. P. Lü, K. Alfaro-Bittner, S. Boccaletti, M. Perc, and S. G. Guan, "Collective dynamics of heterogeneously and nonlinearly coupled phase oscillators," Phys. Rev. Res. **3**, 043004 (2021).
[8] G. Saxena, A. Prasad, and R. Ramaswamy, "Amplitude death: The emergence of stationarity in coupled nonlinear systems," Phys. Rep. **521**, 205–228 (2012).
[9] A. Koseska, E. Volkov, and J. Kurths, "Oscillation quenching mechanisms: Amplitude vs oscillation death," Phys. Rep. **531**, 173–199 (2013).
[10] A. Koseska, E. Volkov, and J. Kurths, "Transition from amplitude to oscillation death via Turing bifurcation," Phys. Rev. Lett. **111**, 024103 (2013).
[11] S. De Monte, F. d'Ovidio, and E. Mosekilde, "Coherent regimes of globally coupled dynamical systems," Phys. Rev. Lett. **90**, 054102 (2003).
[12] H. Daido and K. Nakanishi, "Aging transition and universal scaling in oscillator networks," Phys. Rev. Lett. **93**, 104101 (2004).
[13] W. Zou, D. V. Senthilkumar, M. Zhan, and J. Kurths, "Quenching, aging, and reviving in coupled dynamical networks," Phys. Rep. **931**, 1–72 (2021).
[14] A. Jenkins, "Self-oscillation," Phys. Rep. **525**, 167–222 (2013).
[15] W. Zou, C. G. Yao, and M. Zhan, "Eliminating delay-induced oscillation death by gradient coupling," Phys. Rev. E **82**, 056203 (2010).
[16] W. Zou, D. V. Senthilkumar, M. Zhan, and J. Kurths, "Reviving oscillations in coupled nonlinear oscillators," Phys. Rev. Lett. **111**, 014101 (2013).
[17] W. Zou, D. V. Senthilkumar, R. Nagao, I. Z. Kiss, Y. Tang, A. Koseska, J. Duan, and J. Kurths, "Restoration of rhythmicity in diffusively coupled dynamical networks," Nat. Commun. **6**, 7709 (2015).
[18] V. K. Chandrasekar, S. Karthiga, and M. Lakshmanan, "Feedback as a mechanism for the resurrection of oscillations from death states," Phys. Rev. E **92**, 012903 (2015).
[19] W. Zou, M. Zhan, and J. Kurths, "Revoking amplitude and oscillation deaths by low-pass filter in coupled oscillators," Phys. Rev. E **95**, 062206 (2017).
[20] H.-A. Tanaka, A. J. Lichtenberg, and S. Oishi, "First order phase transition resulting from finite inertia in coupled oscillator systems," Phys. Rev. Lett. **78**, 2104–2107 (1997).
[21] H.-A. Tanaka, A. J. Lichtenberg, and S. Oishi, "Self-synchronization of coupled oscillators with hysteretic responses," Phys. D **100**, 279–300 (1997).

[22] J. A. Acebrón and R. Spigler, “Adaptive frequency model for phase-frequency synchronization in large populations of globally coupled nonlinear oscillators,” Phys. Rev. Lett. **81**, 2229–2232 (1998).
[23] J. A. Acebrón and R. Spigler, “Uncertainty in phase-frequency synchronization of large populations of globally coupled nonlinear oscillators,” Phys. D **141**, 65–79 (2000).
[24] J. A. Acebrón, L. L. Bonilla, and R. Spigler, “Synchronization in populations of globally coupled oscillators with inertial effects,” Phys. Rev. E **62**, 3437–3454 (2000).
[25] S. Gupta, A. Campa, and S. Ruffo, “Nonequilibrium first-order phase transition in coupled oscillator systems with inertia and noise,” Phys. Rev. E **89**, 022123 (2014).
[26] J. Barré and D. Métivier, “Bifurcations and singularities for coupled oscillators with inertia and frustration,” Phys. Rev. Lett. **117**, 214102 (2016).
[27] B. Ermentrout, “An adaptive model for synchrony in the firefly *Pteroptyx malaccae*,” J. Math. Biol. **29**, 571–585 (1991).
[28] K. Wiesenfeld, P. Colet, and S. H. Strogatz, “Synchronization transitions in a disordered Josephson series array,” Phys. Rev. Lett. **76**, 404–407 (1996).
[29] B. Trees, V. Saranathan, and D. Stroud, “Synchronization in disordered Josephson junction arrays: Small-world connections and the Kuramoto model,” Phys. Rev. E **71**, 016215 (2005).
[30] G. Filatrella, A. H. Nielsen, and N. F. Pedersen, “Analysis of a power grid using a Kuramoto-like mode,” Eur. Phys. J. B **61**, 485–491 (2008).
[31] S. Olmi, A. Navas, S. Boccaletti, and A. Torcini, “Hysteretic transitions in the Kuramoto model with inertia,” Phys. Rev. E **90**, 042905 (2014).
[32] D. Métivier, L. Wetzel, and S. Gupta, “Onset of synchronization in networks of second-order Kuramoto oscillators with delayed coupling: Exact results and application to phase-locked loops,” Phys. Rev. Res. **2**, 023183 (2020).
[33] Y. Braiman, J. F. Lindner, and W. L. Ditto, “Taming spatiotemporal chaos with disorder,” Nature (London) **378**, 465–467 (1995).
[34] M. Weiss, T. Kottos, and T. Geisel, “Taming chaos by impurities in two-dimensional oscillator arrays,” Phys. Rev. E **63**, 056211 (2001).
[35] F. Qi, Z. H. Hou, and H. W. Xin, “Ordering chaos by random shortcuts,” Phys. Rev. Lett. **91**, 064102 (2003).
[36] S. F. Brandt, B. K. Dellen, and R. Wessel, “Synchronization from disordered driving forces in arrays of coupled oscillators,” Phys. Rev. Lett. **96**, 034104 (2006).
[37] R. Chacón and P. J. Martínez, “Controlling chaotic solitons in Frenkel-Kontorova chains by disordered driving forces,” Phys. Rev. Lett. **98**, 224102 (2007).
[38] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort, and R. Spigler, “The Kuramoto model: A simple paradigm for synchronization phenomena,” Rev. Mod. Phys. **77**, 137–185 (2005).
[39] D. G. Aronson, G. B. Ermentrout, and N. Kopell, “Amplitude response of coupled oscillators,” Phys. D **41**, 403–449 (1990).
[40] D. V. Ramana Reddy, A. Sen, and G. L. Johnston, “Time delay induced death in coupled limit-cycle oscillators,” Phys. Rev. Lett. **80**, 5109–5112 (1998).
[41] D. V. Ramana Reddy, A. Sen, and G. L. Johnston, “Time delay effects on coupled limit cycle oscillators at Hopf bifurcation,” Phys. D **129**, 15–34 (1999).
[42] S. H. Strogatz, “Death by delay,” Nature **394**, 316–317 (1998).
[43] K. Konishi, “Amplitude death induced by dynamic coupling,” Phys. Rev. E **68**, 067202 (2003).
[44] F. M. Atay, “Distributed delays facilitate amplitude death of coupled oscillators,” Phys. Rev. Lett. **91**, 094101 (2003).
[45] R. Karnatak, R. Ramaswamy, and A. Prasad, “Amplitude death in the absence of time delays in identical coupled oscillators,” Phys. Rev. E **76**, 035201(R) (2007).
[46] A. Sharma and M. D. Shrimali, “Amplitude death with mean-field diffusion,” Phys. Rev. E **85**, 057204 (2012).
[47] K. Bar-Eli, “On the stability of coupled chemical oscillators,” Phys. D **14**, 242–252 (1985).
[48] P. C. Matthews and S. H. Strogatz, “Phase diagram for the collective behavior of limit-cycle oscillators,” Phys. Rev. Lett. **65**, 1701–1704 (1990).
[49] P. C. Matthews, R. E. Mirollo, and S. H. Strogatz, “Dynamics of a large system of coupled nonlinear oscillators,” Phys. D **52**, 293–331 (1991).
[50] R. E. Mirollo and S. H. Strogatz, “Amplitude death in an array of limit-cycle oscillators,” J. Stat. Phys. **60**, 245–262 (1990).
[51] G. B. Ermentrout, “Oscillator death in populations of ‘all to all’ coupled nonlinear oscillators,” Phys. D **41**, 219–231 (1990).
[52] M. Shiino and M. Frankowicz, “Synchronization of infinitely many coupled limit-cycle type oscillators,” Phys. Lett. A **136**, 103–108 (1989).
[53] F. M. Atay, “Oscillator death in coupled functional differential equations near Hopf bifurcation,” J. Differ. Equ. **221**, 190–209 (2006).
[54] F. Sorrentino, L. M. Pecora, A. M. Hagerstrom, T. E. Murphy, and R. Roy, “Complete characterization of the stability of cluster synchronization in complex dynamical networks,” Sci. Adv. **2**, e1501737 (2016).
[55] C. I. del Genio, J. Gómez-Gardeñes, I. Bonamassa, and S. Boccaletti, “Synchronization in networks with multiple interaction layers,” Sci. Adv. **2**, e1601679 (2016).
[56] T. R. Hsu, *Applied Engineering Analysis* (John Wiley, 2018).
[57] F. Parastesh, S. Jafari, H. Azarnoush, Z. Shahriari, Z. Wang, S. Boccaletti, and M. Perc, “Chimeras,” Phys. Rep. **898**, 1–114 (2021).
[58] M. H. Matheny, J. Emenheiser, W. Fon, A. Chapman, A. Salova, M. Rohden, J. Li, M. H. de Badyn, M. Pósfai, L. Duenas-Osorio, M. Mesbahi, J. P. Crutchfield, M. C. Cross, R. M. D’Souza, and M. L. Roukes, “Exotic states in a simple network of nanoelectromechanical oscillators,” Science **363**, eaav7932 (2019).
[59] D. V. R. Reddy, A. Sen, and G. L. Johnston, “Experimental evidence of time-delay-induced death in coupled limit-cycle oscillators,” Phys. Rev. Lett. **85**, 3381–3384 (2000).
[60] I. Ozden, S. Venkataramani, M. A. Long, B. W. Connors, and A. V. Nurmikko, “Strong coupling of nonlinear electronic and biological oscillators: Reaching the amplitude death regime,” Phys. Rev. Lett. **93**, 158102 (2004).
[61] Y. Zhai, I. Z. Kiss, and J. L. Hudson, “Amplitude death through a Hopf bifurcation in coupled electrochemical oscillators: Experiments and simulations,” Phys. Rev. E **69**, 026208 (2004).
[62] H. Hyodo, M. Iwasaki, and T. Biwa, “Suppression of Rijke tube oscillations by delay coupling,” J. Appl. Phys. **128**, 094902 (2020).